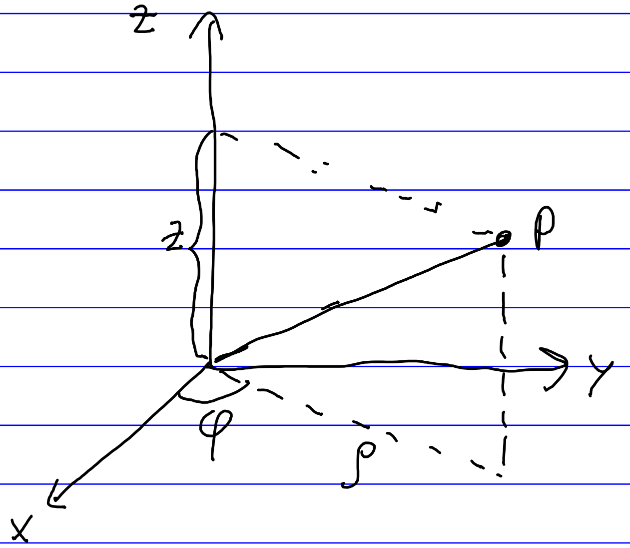
Solving the Laplace eq. in cylindrical coords

Cylindrical coord. system : (ρ, φ, z)



$P : (\rho, \varphi, z)$

Laplacian in cylindrical coords:

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2}$$

The ρ derivative can be rewritten

$$\frac{1}{\rho}\left(\rho \frac{\partial^2 V}{\partial \rho^2} + \frac{\partial V}{\partial \rho}\right) = \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho}$$

$\Rightarrow$ Laplace eq $\nabla^2 V = 0$ reads

$$\frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Look for separable solutions, i.e.

$$V(\rho,\varphi,z) \equiv R(\rho)\,\Phi(\varphi)\,Z(z)$$

Inserting into LE and dividing by $V = R\Phi Z$

$$\Rightarrow \underbrace{\frac{1}{R}\left(\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho}\right)}_{\equiv\, h(\rho)} + \underbrace{\frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\varphi^2}}_{\equiv\, g(\varphi) = \text{const.} \equiv -\nu^2} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{\equiv f(z) = \text{const.} \equiv k^2} = 0$$

$$\Rightarrow \frac{1}{R}\left(\frac{\partial^2 R}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial R}{\partial\rho}\right) + \frac{(-\nu^2)}{\rho^2} + k^2 = 0$$

$$\Rightarrow \boxed{\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right)R = 0}$$ eq. for $R(\rho)$

$$\boxed{\frac{d^2\Phi}{d\varphi^2} + \nu^2\Phi = 0}$$ eq. for $\Phi(\varphi)$

$$\boxed{\frac{d^2Z}{dz^2} - k^2 Z = 0}$$ eq. for $Z(z)$

However, in this coord system the arguments for why, say, $g(\varphi)$ must be a constant, seem less straightforward, so let us look at it. We have

$$h(\rho) + \frac{1}{\rho^2}\,g(\varphi) + f(z) = 0$$

Keeping ρ and φ fixed, and varying z, requires $f(z) = \text{const.}$ for the LHS to remain 0.

So set $f(z) \equiv k^2$

$$\Rightarrow \quad h(\rho) + \frac{1}{\rho^2} g(\varphi) + k^2 = 0$$

$$\Rightarrow \quad \underbrace{\rho^2 \left(h(\rho) + k^2 \right)}_{\text{only func. of } \rho} + g(\varphi) = 0$$

Keeping ρ fixed and varying φ, requires $g(\varphi) = \text{const.}$ for LHS to remain 0.

So set $g(\varphi) \equiv -\nu^2$

$$\Rightarrow \quad \rho^2 \left(h(\rho) + k^2 \right) - \nu^2 = 0$$

$$\Rightarrow \quad h(\rho) + k^2 - \frac{\nu^2}{\rho^2} = 0 \quad \Rightarrow \text{ eq. for } R(\rho)$$

Let us now assume k real (and positive)
$\Rightarrow$ k^2 real and positive

$$\Rightarrow Z(z) = A e^{kz} + B e^{-kz} \qquad \text{(exp. increasing/decreasing)}$$

We also assume ν real (and positive)
$\Rightarrow$ ν^2 real and positive

$$\Rightarrow \quad \Phi(\varphi) = C \cos(\nu\varphi) + D \sin(\nu\varphi)$$

$$= C' e^{i\nu\varphi} + D' e^{-i\nu\varphi}$$

(oscillating, in line with periodic nature of φ:
$\Phi(\varphi + 2\pi) = \Phi(\varphi)$)
$\Downarrow$
ν integer

Radial eq: Let $x \equiv k\rho$ where ρ is radial coord and k is the constant parameter

$$\Rightarrow \quad \frac{d}{d\rho} = \frac{dx}{d\rho}\frac{d}{dx} = k\frac{d}{dx}$$

$$\Rightarrow \quad \frac{d^2}{d\rho^2} = \frac{d}{d\rho}\frac{d}{d\rho} = k^2\frac{d^2}{dx^2}$$

$\Rightarrow$ radial eq $\quad \frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right)R = 0$

$$\cancel{k^2}\frac{d^2R}{dx^2} + \frac{\cancel{k}}{x}\cancel{k}\frac{dR}{dx} + \left(\cancel{k^2} - \nu^2\frac{\cancel{k^2}}{x^2}\right)R = 0$$

$$\Rightarrow \quad \boxed{\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)R = 0}$$ Bessel equation

Solutions are: Bessel functions of order ν ($J_\nu(x)$ and $Y_\nu(x)$)

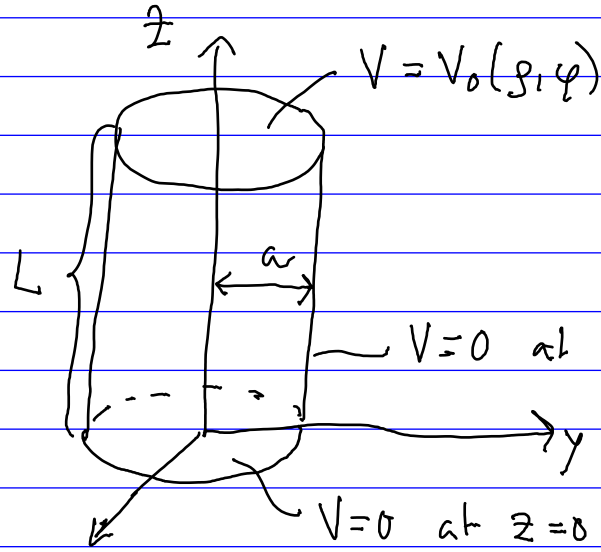
(recall ν is an integer)

$J_\nu(x)$: Bessel function of the first kind
$Y_\nu(x)$: ——— " ——— second kind
($Y_\nu(x)$ is denoted $N_\nu(x)$ (Neumann function) by Jackson)

For a given ν, $J_\nu(x)$ and $Y_\nu(x)$ are linearly indep. solutions of the Bessel eq.

The $Y_\nu(x)$ diverge as $x = k\rho \to 0$, ie. at the z-axis. Considering problems with V finite there, $Y_\nu(x)$ is not permissible $\Rightarrow$ focus on $J_\nu(x)$ only

A concrete example : cylinder



$V = V_0(\rho, \varphi)$ at $z = L$ (iii)

$V = 0$ at $\rho = a$ (ii)

$V = 0$ at $z = 0$ (i)

Find V inside cylinder

Separable sol : $V(\rho, \varphi, z) = R_{k\nu}(\rho)\, \Phi_\nu(\varphi)\, Z_k(z)$

$$= J_\nu(k\rho)\left(C \cos\nu\varphi + D \sin\nu\varphi\right)\left(A e^{kz} + B e^{-kz}\right)$$

(i) $0 = V(\rho, \varphi, z=0) \propto A + B \quad \Rightarrow \quad B = -A$

$$\Rightarrow \quad Z(z) \propto \sinh kz$$

(ii) $0 = V(\rho = a, \varphi, z) \propto J_\nu(ka)$

$$\Rightarrow \quad J_\nu(ka) = 0 \quad \Rightarrow \quad k = k_{\nu n} = \frac{x_{\nu n}}{a}$$

where $x_{\nu n}$ is the n'th root of $J_\nu(x)$
($n = 1, 2, 3, \ldots$) (∞ no. of roots)

$\Rightarrow$ expansion

$\{\sqrt{\rho}\, J_\nu(k_{\nu n}\rho) \mid n = 1, 2, 3 \ldots\}$ (ν fixed) forms complete orthog. set on $0 \le \rho \le a$

$$V(\rho, \varphi, z) = \sum_{\nu=0}^{\infty} \sum_{n=1}^{\infty} J_\nu(k_{\nu n}\rho) \sinh(k_{\nu n} z)\left(C_{\nu n} \cos\nu\varphi + D_{\nu n} \sin m\varphi\right)$$

Can extract coeffs $C_{\nu n}$, $D_{\nu n}$ using orthogonality rels for Bessel functions (ρ-dep) and Fourier series (φ-dep)