Problem 2

(a) Since the potential on the sphere surface is independent of the azimuthal angle φ, the problem has azimuthal symmetry. We can therefore expand the potential as

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

Since $V(r,\theta)$ should $\to 0$ as $r \to \infty$, A_l must be 0 for all l (including $l=0$)

$$\Rightarrow V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \qquad (*)$$

To find the coefficients B_l, we need to consider the expansion $(*)$ for $r=R$ and use that it should equal $V(R,\theta) = V_0 \cos^2\theta$. The coefficients could then be determined by using the orthogonality relations for Legendre polynomials. However, because in this example $V(R,\theta) = V_0 \cos^2\theta$ has a simple expression in terms of Legendre polynomials, we can use that to simply read off the coefficients. To this end, we note that

$$P_0(\cos\theta) = 1 \quad , \quad P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$$

$$\Rightarrow \cos^2\theta - \frac{2}{3} P_2(\cos\theta) = \frac{1}{3} = \frac{1}{3} P_0(\cos\theta)$$

$$\Rightarrow V(R,\theta) = \frac{V_0}{3} \left[P_0(\cos\theta) + 2\, P_2(\cos\theta) \right]$$

Comparing this with (*) evaluated at $r=R$, we can read off

$l=0$ term: $\frac{B_0}{R} = \frac{V_0}{3} \Rightarrow B_0 = \frac{V_0 R}{3}$

$l=2$ term: $\frac{B_2}{R^3} = \frac{2V_0}{3} \Rightarrow B_2 = \frac{2V_0 R^3}{3}$

$l \neq 0,2$: $\frac{B_l}{R^{l+1}} = 0 \Rightarrow B_l = 0$

Inserting these results back into (*) gives

$$V(r,\theta) = \frac{V_0}{3}\left[\frac{R}{r}\underbrace{P_0(\cos\theta)}_{=1} + 2\left(\frac{R}{r}\right)^3 P_2(\cos\theta)\right]$$

Hence the potential outside the sphere $(r \geq R)$ is

$$\underline{\underline{V_{outside}(r,\theta) = \frac{V_0}{3}\left[\frac{R}{r} + 2\left(\frac{R}{r}\right)^3 P_2(\cos\theta)\right]}}$$

Let us check that this expression reduces to the correct result at $r=R$:

$$V_{outside}(R,\theta) = \frac{V_0}{3}\left[1 + 2P_2(\cos\theta)\right]$$

$$= \frac{V_0}{3}\left[1 + 2\cdot\frac{1}{2}(3\cos^2\theta - 1)\right] = \underline{V_0\cos^2\theta} \quad \text{ok}$$

(b) The surface charge density σ on the spherical surface $r = R$ is given by

$$\sigma = -\epsilon_0 \left[\left.\frac{\partial V_{outside}}{\partial n}\right|_{r=R} - \left.\frac{\partial V_{inside}}{\partial n}\right|_{r=R} \right]$$

On the spherical surface, $\hat{n} = \hat{r}$, so the normal derivative is

$$\frac{\partial}{\partial n} \equiv \hat{n} \cdot \nabla = \hat{r} \cdot \nabla$$

expression for ∇ in spherical coordinates

$$= \hat{r} \cdot \left[\hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\varphi} \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi} \right]$$

$$= \frac{\partial}{\partial r} \qquad (\text{since } \hat{r} \cdot \hat{\theta} = \hat{r} \cdot \hat{\varphi} = 0)$$

$$\Rightarrow \quad \sigma = -\epsilon_0 \left[\left.\frac{\partial V_{outside}}{\partial r}\right|_{r=R} - \left.\frac{\partial V_{inside}}{\partial r}\right|_{r=R} \right]$$

Since we have found $V_{outside}(r, \theta)$, we can find the first term. What about the second term? If the sphere had been a conductor, it would have been 0, since the electric field $E = 0$ in a conductor. But the sphere is not a conductor, because $V(R, \theta)$ is not a constant (it depends on θ). So in order to calculate the second term we first have to find $V(r, \theta)$ inside the sphere.

The azimuthal symmetry again allows us to write

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

B_ℓ must be 0 for all ℓ; otherwise $V(r,\theta)$ would diverge as $r \to 0$

$$\Rightarrow \quad V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell \, r^\ell \, P_\ell(\cos\theta) \qquad (**)$$

Evaluating this at $r=R$ and comparing with $V(R,\theta) = \frac{V_0}{3}\left[P_0(\cos\theta) + 2P_2(\cos\theta)\right]$ gives

$\ell=0$ term: $A_0 R^0 = \frac{V_0}{3} \Rightarrow A_0 = \frac{V_0}{3}$

$\ell=2$ term: $A_2 R^2 = \frac{V_0}{3}\cdot 2 \Rightarrow A_2 = \frac{2V_0}{3R^2}$

$\ell \neq 0,2$: $A_\ell R^\ell = 0 \Rightarrow A_\ell = 0$

Inserting these results back into $(**)$ gives

$$V_{inside}(r,\theta) = \frac{V_0}{3}\left[P_0(\cos\theta) + 2\left(\frac{r}{R}\right)^2 P_2(\cos\theta)\right]$$

(One can see that it reduces to $V_0 \cos^2\theta$ at $r=R$, as it should.)

To summarize, we have found

$$V_{outside}(r,\theta) = \frac{V_0}{3}\left[\frac{R}{r} + 2\left(\frac{R}{r}\right)^3 P_2(\cos\theta)\right]$$

$$V_{inside}(r,\theta) = \frac{V_0}{3}\left[1 + 2\left(\frac{r}{R}\right)^2 P_2(\cos\theta)\right]$$

This gives

$$\frac{\partial V_{outside}}{\partial r} = \frac{V_0}{3}\left[-\frac{R}{r^2} + 2\cdot 3\left(\frac{R}{r}\right)^2 \cdot \left(\frac{-R}{r^2}\right) P_2(\cos\theta)\right]$$

$$\Rightarrow \left.\frac{\partial V_{outside}}{\partial r}\right|_{r=R} = -\frac{V_0}{3R}\left[1 + 6P_2(\cos\theta)\right]$$

$$\frac{\partial V_{inside}}{\partial r} = \frac{V_0}{3} \cdot 2 \cdot 2 \frac{r}{R} \cdot \frac{1}{R} P_2(\cos\theta)$$

$$\Rightarrow \left.\frac{\partial V_{inside}}{\partial r}\right|_{r=R} = \frac{4V_0}{3R} P_2(\cos\theta)$$

The surface charge density is therefore

$$\sigma = -\epsilon_0 \frac{V_0}{3R} \left\{ -(1 + 6P_2(\cos\theta)) - 4P_2(\cos\theta) \right\}$$

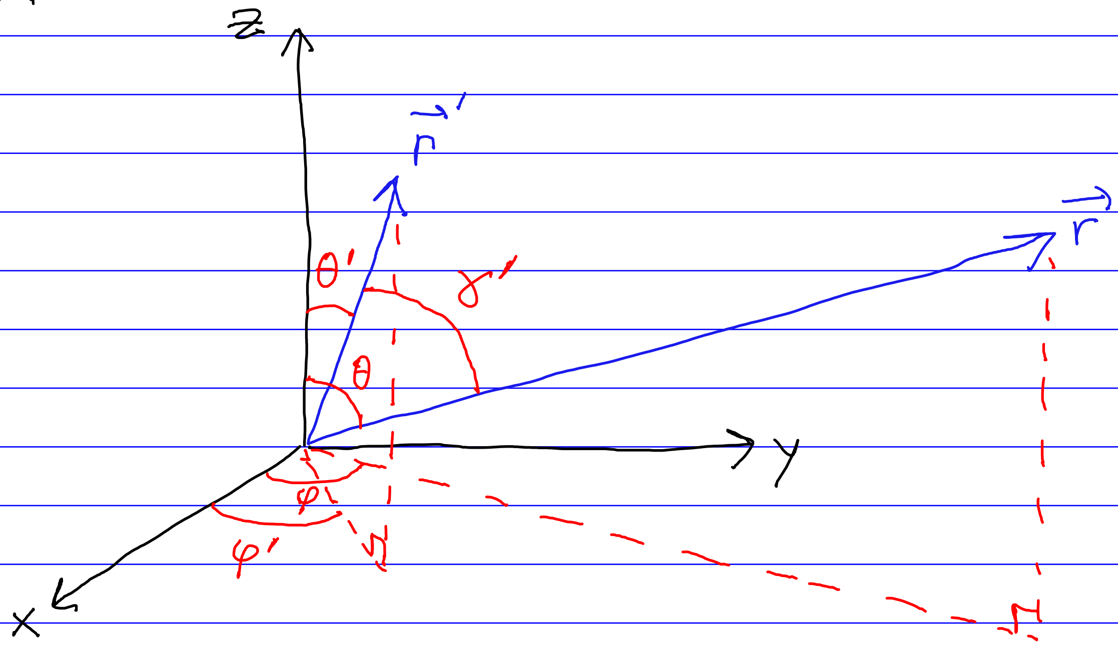
$$= \underline{\underline{\frac{\epsilon_0 V_0}{3R}\left[1 + 10 P_2(\cos\theta)\right] \equiv \sigma(\theta)}}$$

(c) In (a) we found $V_{outside}(r,\theta)$ from the potential $V(R,\theta)$ on the spherical surface. Now we are instead asked to find $V_{outside}(r,\theta)$ from the charge distribution (given by the surface charge density $\sigma(\theta)$ on the spherical surface). More precisely we are asked to find the <u>quadrupole term</u> in the multipole expansion for $V_{outside}$. We know this should be nonzero since there is a quadrupole contribution (i.e. $\propto 1/r^3$) in the expression we found for $V_{outside}$. (There is also a monopole contribution (i.e. $\propto 1/r$) which we could also alternatively find from the multipole expansion).

The quadrupole term in the multipole expansion is

$$V_{quad}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int d^3r'\, \rho(\vec{r}')(r')^2 P_2(\cos\gamma')$$

where γ' is the angle between $\vec{r}$ and $\vec{r}'$. In Griffiths and in lectures this angle was called θ', but I now wish to use θ' for the angle between $\hat{r}'$ and $\hat{z}$; see the figure.



Consider the integral in the quadrupole term. Using the expression for P_2, it reads

$$\frac{1}{2} \int d^3r' \, (r')^2 \rho(\vec{r}') \left(3\cos^2\gamma' - 1 \right) \qquad (\square)$$

Since

$$\cos\gamma' = \hat{r}' \cdot \hat{r} = \hat{r}_i \hat{r}_i' \qquad \text{(summation convention)}$$

we get $\cos^2\gamma' = \hat{r}_i \hat{r}_j \hat{r}_i' \hat{r}_j'$,

so $(\square)$ can be written (cf. Problem 3.45 in Griffiths)

$$\frac{1}{2} \left[3 \hat{r}_i \hat{r}_j \int d^3r' \, (r')^2 \rho(\vec{r}') \, \hat{r}_i' \hat{r}_j' - \int d^3r' \, (r')^2 \rho(\vec{r}') \right]$$

Note that by rewriting in this way, all integrals are now independent of $\vec{r}$.

The charge density is given by

$$\rho(\vec{r}\,') = \sigma(\theta')\,\delta(r'-R).$$

Thus

$$\int d^3r'\,(r')^2\,\rho(\vec{r}\,') = \int \underbrace{dr'(r')^2\,d\varphi'\,\sin\theta'\,d\theta'}_{d^3r'}(r')^2\,\delta(r'-R)\,\sigma(\theta')$$

from $\int d\varphi'$

$$= 2\pi\,R^4\int_0^\pi d\theta'\,\sin\theta'\,\underbrace{\sigma(\theta')}\quad \leftarrow = \frac{\epsilon_0 V_0}{3R}\left[P_0(\cos\theta') + 10P_2(\cos\theta')\right]$$

$$= 2\pi R^4\frac{\epsilon_0 V_0}{3R}\int_0^\pi d\theta'\,\sin\theta'\left[P_0(\cos\theta') + 10\,P_2(\cos\theta')\right]$$

$$= \frac{2\pi\epsilon_0 V_0 R^3}{3}\cdot\int_{-1}^{1} dx\left[1 + 10\,\frac{1}{2}\left(3x^2-1\right)\right]$$

$$= \frac{2\pi\epsilon_0 V_0 R^3}{3}\left[2 + 15\,\frac{1}{3}\underbrace{x^3\Big|_{-1}^{1}}_{2} - 5\cdot 2\right] = \underline{\frac{4\pi\epsilon_0 V_0 R^3}{3}}$$

(As a check, note that this integral can be rewritten $R^2\int d^3r'\,\rho(\vec{r}\,')$ where $\int d^3r'\,\rho(\vec{r}\,')$ is the total charge in the monopole term ($\propto 1/r$) in $V_{outside}$. From this term we can read off the total charge as $4\pi\epsilon_0\frac{V_0 R}{3}$, agreeing with our result above)

Next consider the sum

$$\hat{r}_i \hat{r}_j \int d^3r' (r')^2 \rho(\vec{r}') \hat{r}_i' \hat{r}_j'$$

where
$$\hat{r}_x = \sin\theta \cos\varphi \,, \quad \hat{r}_x' = \sin\theta' \cos\varphi'$$
$$\hat{r}_y = \sin\theta \sin\varphi \,, \quad \hat{r}_y' = \sin\theta' \sin\varphi'$$
$$\hat{r}_z = \cos\theta \,, \quad \hat{r}_z' = \cos\theta'$$

There are $3 \cdot 3 = 9$ terms in the sum. Symbolically we can write the sum as (first letter i, second j)

$$\begin{aligned} & xx + yy + zz + xy + yx \\ + & zx + zy + xz + yz \end{aligned}$$

We note that $\rho(\vec{r}')$ is independent of φ'

$$\Rightarrow \quad zx,\ xz \;\propto\; \int_0^{2\pi} d\varphi' \cos\varphi' = 0$$

$$zy,\ yz \;\propto\; \int_0^{2\pi} d\varphi' \sin\varphi' = 0$$

$$xy,\ yx \;\propto\; \int_0^{2\pi} d\varphi' \cos\varphi' \sin\varphi' = 0$$

Furthermore (define the shorthand $K \equiv d^3r'(r')^2\rho(\vec{r}')$)

$$\begin{aligned} xx + yy = & \sin^2\theta \cos^2\varphi \int K \sin^2\theta' \cos^2\varphi' \\ & + \sin^2\theta \sin^2\varphi \int K \sin^2\theta' \sin^2\varphi' \end{aligned}$$

Since the average of $\cos^2\varphi'$ and $\sin^2\varphi'$ in $[0, 2\pi)$ is $\frac{1}{2}$, we can replace both factors by $\frac{1}{2}$

$$\begin{aligned} \Rightarrow \quad xx + yy &= \frac{1}{2} \sin^2\theta \overbrace{(\cos^2\varphi + \sin^2\varphi)}^{=1} \int K \sin^2\theta' \\ &= \frac{1}{2} \sin^2\theta \int K \sin^2\theta' \end{aligned}$$

$$= \frac{1}{2}(1-\cos^2\theta)\int K\,(1-\cos^2\theta')$$

Finally, $\quad zz = \cos^2\theta \int K \cos^2\theta'$

So the sum reduces to

$$\frac{1}{2}(1-\cos^2\theta)\int K(1-\cos^2\theta') + \cos^2\theta \int K\cos^2\theta'$$

$$= \frac{1}{2}\left(\int K - \int K\cos^2\theta'\right)$$

$$+ \cos^2\theta\left(-\frac{1}{2}\int K + \frac{3}{2}\int K\cos^2\theta'\right)$$

We already calculated $\int K = \frac{4\pi\epsilon_0 V_0 R^3}{3} \equiv C$
so we just need to calculate

$$\int K\cos^2\theta' = \int d^3r'\,(r')^2\,\rho(\vec{r}\,')\cos^2\theta'$$

$$= \int dr'\,(r')^2 \sin\theta'\,d\theta'\,d\varphi'\,(r')^2\,\frac{\epsilon_0 V_0}{3R}\,\delta(r'-R)$$

$$\cdot\left(P_0(\cos\theta') + 10\,P_2(\cos\theta')\right)\underbrace{\left(\frac{1}{3}P_0(\cos\theta') + \frac{2}{3}P_2(\cos\theta')\right)}_{\cos^2\theta'}$$

$$= 2\pi\,\frac{\epsilon_0 V_0}{3R}\cdot R^4\cdot\frac{1}{3}\int_{-1}^{1} dx\,(P_0 + 10P_2)(P_0 + 2P_2)$$

$$= \frac{2\pi}{9}\epsilon_0 V_0 R^3\left[\frac{2}{2\cdot 0+1} + 10\cdot 2\cdot\frac{2}{2\cdot 2+1}\right] \leftarrow \text{using orthogonality relations for Legendre polynomials}$$

$$= \frac{4\pi}{9}\epsilon_0 V_0 R^3\,[1+4] = \frac{4\pi\epsilon_0 V_0 R^3}{3}\cdot\frac{5}{3} = \frac{5}{3}C$$

Thus the sum becomes

$$\frac{1}{2}C\left(1-\frac{5}{3}\right) + \frac{1}{2}\cos^2\theta\left(-C + 3\cdot\frac{5}{3}C\right) = \frac{C}{3}\left(6\cos^2\theta - 1\right)$$

Putting it all together, the quadrupole term becomes

$$\frac{1}{4\pi\epsilon_0}\frac{1}{2r^3}\left[3\cdot\frac{C}{3}\left(6\cos^2\theta-1\right)-C\right]$$

$$=\frac{1}{4\pi\epsilon_0}\frac{C}{r^3}\underbrace{\left(3\cos^2\theta-1\right)}_{2P_2(\cos\theta)}=\frac{1}{\cancel{4\pi\epsilon_0}}\frac{\cancel{4\pi\epsilon_0}V_0}{3}\left(\frac{R}{r}\right)^3 2P_2(\cos\theta)$$

$$=\frac{V_0}{3}\cdot 2\left(\frac{R}{r}\right)^2 P_2(\cos\theta)$$

which indeed agrees with the quadrupole contribution to $V_{outside}$ found in (a).

For the interested reader, I also present an alternative calculation that is more elegant and powerful (involving the full multipole expansion), but invokes a result involving spherical harmonics that we have not proved. It starts from the multipole expansion

$$V(\vec{r})=\frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\int d^3r'(r')^l\rho(\vec{r}')P_l(\cos\gamma')$$

Now use the result ("addition theorem for spherical harmonics")

$$P_l(\cos\gamma')=\frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y^*_{lm}(\theta',\varphi')Y_{lm}(\theta,\varphi)$$

to get

$$V(\vec{r})=\frac{1}{\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y_{lm}(\theta,\varphi)}{(2l+1)\,r^{l+1}}\int d^3r'(r')^l\rho(\vec{r}')Y^*_{lm}(\theta',\varphi')$$

Next use that $Y_{lm}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)\, e^{im\varphi}$

Since our $\rho(\vec{r}\,')$ is indep. of φ', the integral $\int d^3r'$ will be proportional to $\int_0^{2\pi} d\varphi' \exp(-im\varphi') = 2\pi\,\delta_{m,0}$. Thus only the $m=0$ term survives the integration, giving

$$V(\vec{r}) = \frac{1}{\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\frac{Y_{l0}(\theta,\varphi)}{r^{l+1}}\int d^3r'(r')^l \rho(\vec{r}\,')\, Y_{l0}^*(\theta',\varphi')$$

Now use $Y_{l0}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}}\, P_l^0(\cos\theta)$

and $P_l^0(\cos\theta) = P_l(\cos\theta)$, which gives

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{P_l(\cos\theta)}{r^{l+1}}\int d^3r'(r')^l\rho(\vec{r}\,')P_l(\cos\theta')}$$

This expression is valid when $\rho(\vec{r}\,')$ is independent of φ'. Note that although the integral looks superficially like the one discussed in Griffiths and the lectures, the meaning of θ' is different.

Now we will evaluate $V(\vec{r})$ using the above formula for the charge distribution in our case. The integral becomes

$$\int d^3r'(r')^l\rho(\vec{r}\,')\,P_l(\cos\theta')$$

$$= \int dr'(r')^2 d\theta' \sin\theta'\, d\varphi'\,(r')^l\,\frac{\epsilon_0 V_0}{3R}\left(P_0(\cos\theta') + 10P_2(\cos\theta')\right)$$

$$\cdot\,\delta(r'-R)\cdot P_l(\cos\theta')$$

$$= 2\pi R^{l+2}\frac{\epsilon_0 V_0}{3R}\int_{-1}^{1} dx\left(P_0(x) + 10P_2(x)\right)P_l(x)$$

$$= \frac{2\pi\epsilon_0 V_0 R^{l+1}}{3}\left[\frac{2}{2\cdot 0+1}\delta_{l,0} + 10\cdot\frac{2}{2\cdot 2+1}\delta_{l,2}\right]$$

$$= \frac{4\pi\epsilon_0 V_0 R^{l+1}}{3}\left(\delta_{l,0} + 2\delta_{l,2}\right)$$

This gives

$$V(\vec{r}) = \frac{1}{\cancel{4\pi\epsilon_0}}\sum_{l=0}^{\infty}\frac{P_l(\cos\theta)}{r^{l+1}}\cdot\frac{\cancel{4\pi\epsilon_0} V_0 R^{l+1}}{3}\left(\delta_{l,0}+2\delta_{l,2}\right)$$

$$= \frac{V_0}{3}\left[\frac{R}{r} + 2\left(\frac{R}{r}\right)^3 P_2(\cos\theta)\right]$$

which agrees with our result for $V_{outside}$ in (a).