Problem 2.5.5

The rod is at rest in the inertial frame S' and thus moves with velocity v in S. The inverse transformation is given by

$$\begin{aligned} \Delta x &= \gamma(\Delta x' + v\Delta t')\ , && (1)\\ \Delta t &= \gamma(\Delta t' + v\Delta x'/c^2)\ . && (2) \end{aligned}$$

Since the length of an object is defined as difference of coordinates of the ends of the rod at simultaneity, we demand $\Delta t = 0$, i.e.

$$\Delta t' = -v\Delta x'/c^2)\ . \qquad (3)$$

Inserting this expression into Eq.(1), we obtain

$$\begin{aligned} \Delta x &= \gamma(\Delta x' - v^2\Delta x'/c^2))\\ &= \frac{1}{\gamma}\Delta x'\\ &= \frac{L^*}{\gamma}\ , && (4) \end{aligned}$$

where we have used that the length of the rod in S' is its length at rest,i.e. L^*. If we denote Δx (the lenght of the rod in S) by L, the formula tells us that $L < L^*$, i. e. it appears shorter in S than in S'.

Problem 3.6.2

In order to calculate the components of $\tilde{F}^{\mu\nu}$, we need $F_{\mu\nu}$, that is given by

$$F_{\mu\nu} = g_{\mu\alpha}g_{\nu\beta}F^{\alpha\beta}\ . \qquad (5)$$

Specifically, we obtain

$$\begin{aligned} F_{ij} &= g_{\mu i}g_{\nu j}F^{\mu\nu}\\ &= F^{ij}\ , && (6)\\ F_{0i} &= g_{\mu 0}g_{\nu j}F^{\mu\nu}\\ &= -F^{0i}\ , && (7) \end{aligned}$$

This yields

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z\\ -E_x & 0 & -B_z & B_y\\ -E_y & B_z & 0 & -B_x\\ -E_z & -B_y & B_x & 0 \end{pmatrix}\ . \qquad (8)$$

We next need

$$\tilde{F^{\mu\nu}} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \,. \tag{9}$$

For example

$$\begin{aligned}\tilde{F^{01}} &= \frac{1}{2}\epsilon^{01\alpha\beta}F_{\alpha\beta} \\ &= \frac{1}{2}\left(\epsilon^{0123}F_{23} - \epsilon^{0132}F_{32}\right) \\ &= F_{23} \,. \end{aligned} \tag{10}$$

The dual tensor can then be written as

$$\tilde{F}^{\mu\nu} = \underline{\underline{\begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}}} \tag{11}$$

Note the symmetry $\mathbf{E} \to \mathbf{B}$ and $\mathbf{B} \to -\mathbf{E}$. Maxwell's equations with no sources are invariant under this (duality) operation.

Using the matrices (8) and (11), we easily find

$$\tilde{F}^{\mu\nu}F_{\mu\nu} = \underline{\underline{-4\mathbf{E}\cdot\mathbf{B}}} \,. \tag{12}$$

Both $F^{\mu\nu}F_{\mu\nu}$ and $\tilde{F}^{\mu\nu}F_{\mu\nu}$ are gauge invariant. The latter is a pseudoscalar under parity. Finally, we find

$$\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} = \underline{\underline{2\left(\mathbf{B}^2 - \mathbf{E}^2\right)}} \,. \tag{13}$$

This can be obtained by direct calculation or by using the duality between **E** and **B**.

Problem 3.6.4, part 1

The energy-momentum tensor is given by

$$\mathcal{T}^{\mu}_{\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^{\mu}_{\nu}\mathcal{L} \,. \tag{14}$$

This yields

$$\mathcal{T}^{\mu}_{\nu} = \bar{\psi}i\gamma^{\mu}\partial_\nu\psi - \delta^{\mu}_{\nu}\mathcal{L} \,. \tag{15}$$

The energy density is given by $\mu = \nu = 0$ and reads

$$\begin{aligned}\mathcal{T}^{0}_{0} &= \bar{\psi}i\gamma^{0}\partial_0\psi - \mathcal{L} \\ &= \underline{\underline{-\bar{\psi}i\gamma^{j}\partial_j\psi + m\bar{\psi}\psi}} \,. \end{aligned} \tag{16}$$

This can also be written as

$$\mathcal{T}_0^0 \;\; = \;\; \underline{\underline{\psi^\dagger \left[\vec{\alpha}\cdot\vec{p} + \beta m\right] \psi}} \;, \tag{17}$$

where we have used $\vec{p} = -i\nabla$, $\gamma^j = \beta\alpha_j$, $\beta = \gamma^0$, and $\bar{\psi} = \psi^\dagger\gamma^0$. This form is familiar.

The momentum density is given by $\mathcal{T}_j^0$ and reads

$$\mathcal{T}_j^0 \;\; = \;\; \underline{\underline{\bar{\psi} i\gamma^0\partial_j\psi}} \;. \tag{18}$$

NOTE that it is common to define the conserved momentum by raising the index $\mathcal{T}^{\nu 0}$. This changes sign when $\nu = j$ and so $\mathcal{T}^{j0} = -\bar{\psi}i\gamma^0\partial^j\psi = \psi^\dagger p^j\psi$.

Current conservation gives

$$\begin{aligned} \partial_\mu\mathcal{T}_\nu^\mu &= \partial_\mu[\bar{\psi}i\gamma^\mu\partial_\nu\psi - \delta_\mu^\nu\mathcal{L}] \\ &= \partial_0\mathcal{T}_0^0 + \partial_j\mathcal{T}_0^i \;, \end{aligned} \tag{19}$$

where Eq. (15) yields

$$\mathcal{T}_0^j \;\; = \;\; \bar{\psi}i\gamma^j\partial_0\psi \;. \tag{20}$$

This gives

$$\partial_\mu\mathcal{T}_0^\mu \;\; = \;\; i(\partial_j\bar{\psi})\gamma^j\partial_0\psi - i(\partial_0\bar{\psi})\gamma^j\partial_j\psi + m\partial_0(\bar{\psi}\psi) \;. \tag{21}$$

We rewrite this as

$$\begin{aligned} \partial_\mu\mathcal{T}_0^\mu &= i(\partial_j\bar{\psi})\gamma^j\partial_0\psi + m\bar{\psi}\partial_0\psi + \partial_0(\bar{\psi})\gamma^0(\partial_0\psi) - i(\partial_0\bar{\psi})\gamma^j\partial_j\psi - \partial_0(\bar{\psi})\gamma^0(\partial_0\psi) + m(\partial_0\bar{\psi})\psi \\ &= \partial_0\psi[(\partial_\mu\bar{\psi})i\gamma^\mu + m\bar{\psi}] - \partial_0\bar{\psi}[i\gamma^\mu(\partial_\mu\psi) - m\psi] \\ &= 0 \;, \end{aligned} \tag{22}$$

since the terms are proportional to the Dirac equation for ψ and $\bar{\psi}$, respectively.