Exercises for week 4

(Just to emphasize what I also said in the lectures: If any of you have questions about tutorial exercises and how to solve them before the Thursday tutorial class, you are very welcome to email me or drop by my office to discuss.)

In the lectures we have focused on cases where the Lagrangian density $\mathcal{L}$ depends on a single field φ and its derivatives $\partial_\mu\varphi$. The generalization to multiple fields is straightforward. If $\mathcal{L}$ depends on N fields φ_i and their derivatives $\partial_\mu\varphi_i$, there is one Euler-Lagrange equation for each field:

$$\frac{\partial\mathcal{L}}{\partial\varphi_i} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)} = 0 \quad (i = 1, \ldots, N). \tag{1}$$

Furthermore, if an infinitesimal transformation of these fields given by

$$\varphi_i(x) \to \varphi_i'(x) = \varphi_i(x) + \beta\Delta\varphi_i(x) \quad (i = 1, \ldots, N) \tag{2}$$

causes the Lagrangian density to transform as

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \beta\partial_\mu\mathcal{J}^\mu, \tag{3}$$

Noether's theorem gives the local conservation law $\partial_\mu j^\mu = 0$ with

$$j^\mu = \sum_i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\Delta\varphi_i - \mathcal{J}^\mu. \tag{4}$$

Exercise 1

Consider the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\sum_{i=1,2}\left[(\partial_\mu\phi_i)(\partial^\mu\phi_i) - m^2\phi_i^2\right] \tag{5}$$

where ϕ_1 and ϕ_2 are real scalar fields.

(a) Find the equations of motion.

(b) Consider the field transformation given by

$$\begin{aligned} \phi_1 \quad &\to \quad \phi_1' = \phi_1\cos\alpha - \phi_2\sin\alpha, & (6)\\ \phi_2 \quad &\to \quad \phi_2' = \phi_1\sin\alpha + \phi_2\cos\alpha. & (7) \end{aligned}$$

You may recognize this as a rotation by angle α in the two-dimensional space with ϕ_1 along the horizontal axis and ϕ_2 along the vertical axis.

Show that the Lagrangian density is invariant under this transformation, i.e. $\mathcal{L}' = \mathcal{L}$.

(c) Find the infinitesimal form of the transformation given by Eqs. (6)-(7).

(d) Show that

$$j^\mu = \phi_1 \partial^\mu \phi_2 - \phi_2 \partial^\mu \phi_1. \tag{8}$$

Exercise 2

The purpose of this exercise is to illustrate how the classical field theory defined by Eq. (5) in terms of the two real fields ϕ_1 and ϕ_2 can alternatively and more simply be described and analyzed in terms of the complex field Φ defined as

$$\Phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \tag{9}$$

and its complex conjugate Φ^*.

(a) Show that the Lagrangian density in Eq. (5) can be written

$$\mathcal{L} = (\partial_\mu \Phi^*)(\partial^\mu \Phi) - m^2 \Phi^* \Phi. \tag{10}$$

(b) It is possible, and very convenient, to derive the equations of motion by treating Φ and Φ^* formally as independent[1] fields to be used in the E-L equations (1). Find the equation of motion resulting from the Euler-Lagrange equation for Φ^*, i.e.[2]

$$\frac{\partial \mathcal{L}}{\partial \Phi^*} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^*)} = 0. \tag{11}$$

Show that the result is indeed equivalent to the equations of motion found in Exercise 1.(a).

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[1]Note that ϕ_1 and ϕ_2 can be expressed in terms of Φ and Φ^* as

$$\begin{aligned} \phi_1 &= \frac{1}{\sqrt{2}}(\Phi + \Phi^*), \\ \phi_2 &= -\frac{i}{\sqrt{2}}(\Phi - \Phi^*). \end{aligned}$$

Treating Φ and Φ^* as if they were independent then gives

$$\begin{aligned} \frac{\partial}{\partial \Phi} &= \frac{\partial \phi_1}{\partial \Phi}\frac{\partial}{\partial \phi_1} + \frac{\partial \phi_2}{\partial \Phi}\frac{\partial}{\partial \phi_2} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial \phi_1} - i\frac{\partial}{\partial \phi_2}\right), \\ \frac{\partial}{\partial \Phi^*} &= \frac{\partial \phi_1}{\partial \Phi^*}\frac{\partial}{\partial \phi_1} + \frac{\partial \phi_2}{\partial \Phi^*}\frac{\partial}{\partial \phi_2} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial \phi_1} + i\frac{\partial}{\partial \phi_2}\right). \end{aligned}$$

From these equations it follows that $\frac{\partial \Phi}{\partial \Phi} = \frac{\partial \Phi^*}{\partial \Phi^*} = 1$ and $\frac{\partial \Phi^*}{\partial \Phi} = \frac{\partial \Phi^*}{\partial \Phi} = 0$, as expected for independent fields.

[2]We could have considered the Euler-Lagrange equation for Φ as well, but this wouldn't give any additional information, as it is just the complex conjugate of Eq. (11).

(c) The transformation (6)-(7) can be expressed in terms of Φ and Φ^* as

$$\Phi \to \Phi' = \Phi\, e^{i\alpha} \quad (\Rightarrow \Phi^* \to \Phi'^* = \Phi^* e^{-i\alpha}). \tag{12}$$

Verify that $\mathcal{L}$, as expressed in Eq. (10), is indeed invariant under this transformation.

(d) Find the infinitesimal form of (12).

(e) By using Φ and Φ^* as the fields in Eq. (4), show that

$$j^\mu = i\left[\Phi\partial^\mu\Phi^* - \Phi^*\partial^\mu\Phi\right]. \tag{13}$$

Verify that this agrees with the expression (8).