Exercise for week 7

In this exercise we will discuss the Dirac equation in **two** spatial dimensions. The spatial coordinates are taken to be $x^1 = x$ and $x^2 = y$. We set $c = 1$ for simplicity, so the time coordinate $x^0 = t$. We also set $\hbar = 1$.

(a) Show that the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \qquad (\mu, \nu = 0, 1, 2) \tag{1}$$

are satisfied by taking

$$\gamma^0 = \sigma_3, \quad \gamma^1 = i\sigma_2, \quad \gamma^2 = -i\sigma_1, \tag{2}$$

where σ_i are the Pauli matrices.

(b) Consider an electromagnetic field represented by $A^\mu = (A^0, A^1, A^2)$. Calculate the electric and magnetic fields $\vec{E}$ and $\vec{B}$ for the following three cases: $A^\mu = (0, 0, Bx)$, $A^\mu = (0, -By, 0)$, and $A^\mu = (0, -\frac{1}{2}By, \frac{1}{2}Bx)$.

(c) We couple the Dirac particle to an external field A^μ. By the standard procedure, this gives the equation

$$(i\gamma^\mu \partial_\mu - q\gamma^\mu A_\mu - m)\psi \;=\; 0, \tag{3}$$

where q is the charge of the particle. Taking $A^\mu = (0, 0, Bx)$ in this equation, use the explicit representation of the Pauli matrices to write it on matrix form.

(d) Explain why it is natural to use the ansatz

$$\psi(\vec{r}, t) \;=\; e^{ip_y y - iEt} \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}. \tag{4}$$

(e) Use the ansatz to derive an equation of the form $Q \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} = 0$, where Q is a 2×2 matrix. Simplify expressions by introducing the operators

$$\xi_\pm \;=\; -i\partial_x \mp i(p_y - qBx). \tag{5}$$

(f) Eliminate the function $g(x)$ from the problem to obtain an equation for $f(x)$ alone.

(g) Solve this equation, thus finding the solutions for both $f(x)$ and E (hint: harmonic oscillator).

(h) Finally, find $g(x)$.