Exercises for week 10

Exercise 1

Problems 5.6.5 and 5.6.6 in JOA's lecture notes.

Exercise 2

In the lectures we considered the interaction-picture time evolution operator

$$U(t,t') = T\left\{\exp\left[-i\int_{t'}^{t} dt''\, H_I(t'')\right]\right\} \qquad (t \geq t') \tag{1}$$

where T is the time-ordering operator and

$$H_I(t) = e^{iH_0(t-t_0)} H_{\text{int}}\, e^{-iH_0(t-t_0)} \tag{2}$$

where t_0 is a reference time. The operator $U(t,t')$ satisfies the first-order differential equation

$$i\frac{\partial}{\partial t}U(t,t') = H_I(t)U(t,t'), \tag{3}$$

which, together with the initial condition

$$U(t',t') = 1, \tag{4}$$

is an equivalent way of specifying $U(t,t')$. We asserted that $U(t,t')$ can be written

$$U(t,t') = e^{iH_0(t-t_0)}e^{-iH(t-t')}e^{-iH_0(t'-t_0)}, \tag{5}$$

which clearly satisfies (4). In the following, use Eq. (5) to

(a) derive the differential equation (3) (thus verifying that the rhs of (5) is a valid expression for $U(t,t')$).

(b) show that $U(t,t')$ is unitary (it is sufficient that you show that $U^\dagger(t,t')U(t,t') = I$.)

(c) show that

$$\begin{aligned} U(t_1,t_2)U(t_2,t_3) &= U(t_1,t_3), \qquad (6) \\ U(t_1,t_3)U^\dagger(t_2,t_3) &= U(t_1,t_2) \qquad (7) \end{aligned}$$

(here $t_1 \geq t_2 \geq t_3$).