Exercises for week 13

Exercise 1

(a) Consider an arbitrary Hermitian single-particle operator $\hat{O}$. In the "1st quantization" formalism, $\hat{O}$ takes the form

$$\hat{O} = \sum_{i=1}^{N} \hat{o}_i \tag{1}$$

where N is the number of particles in the system. Let $\phi_\nu(x)$ and ϵ_ν be the eigenfunctions and associated eigenvalues of $\hat{o}$, i.e.

$$\hat{o}\,\phi_\nu(x) = \epsilon_\nu \phi_\nu(x). \tag{2}$$

Here ν is a set of quantum numbers which completely characterize the single-particle eigenfunctions. As discussed in the lectures, we can use these to construct a basis set for many-particle wavefunctions. The (normalized) basis functions can be written

$$\Phi_{\nu_1,\nu_2,\ldots,\nu_N}(x_1, x_2, \ldots, x_N) = \frac{1}{\sqrt{N!}\sqrt{\prod_\nu n_\nu!}} \sum_{P \in S_N} \zeta^{t_P} \cdot P\phi_{\nu_1}(x_1)\phi_{\nu_2}(x_2)\ldots\phi_{\nu_N}(x_N) \tag{3}$$

where $\xi = \pm 1$ for bosons/fermions and t_P is the number of transpositions (2-particle permutations) associated with the permutation P.[1] S_N is the set of all $N!$ permutations. Furthermore, n_ν is the number of particles in the single-particle state ϕ_ν in the many-particle state $\Phi_{\nu_1,\nu_2,\ldots,\nu_N}$ (for fermions this can only be 0 or 1, hence $\sqrt{\prod_\nu n_\nu!} = 1$ in the fermionic case and can therefore be omitted).

(i) Write down an example of a basis function for a system of 3 fermions where all single-particle states ν_1, ν_2, ν_3 are different (write the state out explicitly, i.e. all $3! = 6$ terms).

(ii) Demonstrate that the function changes sign if the coordinates of two particles are interchanged, e.g. x_1 and x_2. Furthermore, demonstrate that if two or more of the single-particle states are chosen to be the same, the basis function vanishes. These properties are in accordance with the Pauli exclusion principle.

(iii) Show that $\Phi_{\nu_1,\nu_2,\ldots,\nu_N}$ as given in (3) is an eigenfunction of $\hat{O}$ with eigenvalue $\sum_\nu o_\nu n_\nu$.

[1] As discussed in the lectures, t_P is not unique, but its evenness/oddness is, so that the sign ζ^{t_P} is well-defined.

Exercise 2

In the "2nd quantization" formalism, many-particle basis states take the form

$$|n_1, n_2, n_3, \ldots, \rangle = (a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}(a_3^\dagger)^{n_3}\ldots|0\rangle, \tag{4}$$

where $|0\rangle$ is the "vacuum" state with no particles, and $a_\nu^\dagger$ creates a particle in the single-particle state with wavefunction $\phi_\nu(x)$.[2]

(i) Write down the basis state in 2nd quantization that corresponds to the 1st quantization basis state discussed in Exercise 1.(i).

(ii) An explicit connection between the basis states $|n_1, n_2, \ldots\rangle \equiv |n\rangle$ in 2nd quantization and the basis functions $\Phi_n(x_1, x_2, \ldots, x_N)$ in 1st quantization can be established. Let

$$|x_1, x_2, \ldots, x_N\rangle \equiv \frac{1}{\sqrt{N!}}\psi^\dagger(x_1)\psi^\dagger(x_2)\ldots\psi^\dagger(x_N)|0\rangle. \tag{5}$$

Then the (correctly normalized) basis wavefunctions are given by

$$\Phi_n(x_1, \ldots, x_N) = \langle x_1, \ldots, x_N|n\rangle. \tag{6}$$

As an example, take $|n\rangle$ to be the 3-particle fermionic state considered in Exercise 2.(i). Calculate the rhs of Eq. (6) to show that the wavefunction is indeed that considered in Exercise 1.(i).

[2] In Eq. (4) we have for simplicity just replaced the quantum numbers ν labeling a single-particle state by an integer label, i.e. the different states in the single-particle basis are labeled $1, 2, 3, \ldots$.