Exercises for week 16

Exercise 1

Consider the following Hamiltonian describing electrons hopping between nearest-neighbour sites on a two-dimensional square lattice:

$$H = -t \sum_{\langle i,j \rangle} \sum_{\sigma} (c^{\dagger}_{j,\sigma} c_{i,\sigma} + \text{h.c.}). \tag{1}$$

Here $t > 0$ is the hopping amplitude, i and j are labels for the sites of the square lattice, and $\sigma = \pm 1/2$ labels the electron spin projection. The leftmost sum is over all pairs of nearest-neighbour sites (each such pair being counted once).

(a) Show that the Hamiltonian can be written on the diagonalized form

$$H = \sum_{\boldsymbol{k},\sigma} \varepsilon_{\boldsymbol{k}} c^{\dagger}_{\boldsymbol{k},\sigma} c_{\boldsymbol{k},\sigma}, \tag{2}$$

where the sum over $\boldsymbol{k}$ runs over the 1st Brillouin zone of the square lattice. Give the dispersion relation $\varepsilon_{\boldsymbol{k}}$.

(b) Consider the density parameter $n = N_e/N$, where N_e is the number of electrons in the system and N is the number of sites. Show that in the ground state of the system (for a given number N_e of electrons), n is proportional to the $\boldsymbol{k}$-space area enclosed by the Fermi surface and find the proportionality constant.

(c) Sketch the Fermi surface for (i) $n \ll 1$, (ii) $n = 1$, and (iii) $n = 2$.

Exercise 2

(a) Show that an alternative and equivalent form of the spin commutation relations

$$[S^x, S^y] = iS^z, \quad [S^y, S^z] = iS^x, \quad [S^z, S^x] = iS^y \tag{3}$$

(where we have set $\hbar = 1$) is given by

$$[S^+, S^-] = 2S^z, \quad [S^z, S^{\pm}] = \pm S^{\pm}, \tag{4}$$

where $S^{\pm} = S^x \pm iS^y$ are the spin raising and lowering operators.

(b) The Holstein-Primakoff (HP) representation is given by

$$S^+ = \sqrt{2S - \hat{n}}\, a, \tag{5}$$
$$S^- = a^\dagger \sqrt{2S - \hat{n}}, \tag{6}$$
$$S^z = S - \hat{n}, \tag{7}$$

where a and $a^\dagger$ are canonical boson operators, and $\hat{n} = a^\dagger a$. Show that the HP representation satisfies the correct spin commutation relations and the relation $\boldsymbol{S} \cdot \boldsymbol{S} = S(S+1)$.