Part 1

Interacting field theories

Example: "φ^4 theory"
Reference: Peskin & Schroeder, Ch. 4, esp. 4.2

Consider the classical field theory given by

Lagrangian density for φ^4 theory

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{1}{2}m^2\varphi^2}_{\equiv \mathcal{L}_0} \underbrace{- \frac{\lambda}{24}\varphi^4}_{\equiv \mathcal{L}_{int}}$$

Because $\mathcal{L}_{int}$ is independent of $\partial_0\varphi$,
$\pi \equiv \frac{\partial\mathcal{L}}{\partial(\partial_0\varphi)} = \dot{\varphi}$ as for $\mathcal{L}_0$

Hamiltonian density:

$$\mathcal{H} = \pi\dot{\varphi} - \mathcal{L}$$

$$= \underbrace{\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\varphi)^2}_{\equiv \mathcal{H}_0} + \underbrace{\frac{\lambda}{24}\varphi^4}_{\equiv \mathcal{H}_{int}}$$

Hamiltonian:

$$H = \int d^3x\ \mathcal{H} = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{\lambda}{4!}\varphi^4\right]$$

$$\equiv H_0 + H_{int}$$

$$H_{int} = \int d^3x \ \frac{\lambda}{4!} \ \varphi^4(x)$$

The Euler-Lagrange equation (equation of motion) is now nonlinear $\Rightarrow$ the theory cannot be solved exactly.

Consider the quantum field theory obtained by canonical quantization of the classical theory. Because we don't have the set of exact solutions of the classical theory, we cannot expand the field in these, as we did in the $\lambda = 0$ case earlier.

Instead: use perturbation theory, treating λ as "small" $\Rightarrow$ perturbation expansion in λ

H_0: noninteracting particles w/ momentum $\vec{p}$, energy $E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$

H_{int}: will cause scattering of particles

Can use perturbation theory to calculate scattering cross sections and other physically measurable quantities. However, such

quantities are complicated to calculate.
Therefore we will in this course consider a simpler, but also more abstract quantity, the ~~two~~ two-point correlation function / two-point Green's function

$$\langle \Omega | T\{\varphi(x)\,\varphi(y)\} | \Omega \rangle$$

$|\Omega\rangle$: ground state of H

$$\varphi(\vec{x},t) = e^{iH(t-t_0)}\,\varphi(\vec{x},t_0)\,e^{-iH(t-t_0)}$$

(t_0: reference time)

If $\lambda = 0 \Rightarrow H = H_0$ and so the two-point function equals $D_F(x-y)$.
Then the time evolution would be simple:

$$\underbrace{e^{iH_0(t-t_0)}\,\varphi(\vec{x},t_0)\,e^{-iH_0(t-t_0)}}_{\text{time evolution determined by } H_0, \text{ thus simple}} \equiv \varphi_I(\vec{x},t)$$

"interaction picture field"
(what we called $\varphi(\vec{x},t)$ for the free field theory)

Express $\varphi(\vec{x},t)$ in terms of $\varphi_I(\vec{x},t)$:

$$\varphi(\vec{x},t) = e^{iH(t-t_0)}\,\varphi(\vec{x},t_0)\,e^{-iH(t-t_0)}$$

$$= e^{iH(t-t_0)}\,e^{-iH_0(t-t_0)}\,\varphi_I(\vec{x},t)\,e^{iH_0(t-t_0)}\,e^{-iH(t-t_0)}$$

$$\equiv \; U^{+}(t,t_0)\; \varphi_I(\vec{x},t)\; U(t,t_0)$$

where

$$U(t,t_0) \equiv e^{iH_0(t-t_0)}\; e^{-iH(t-t_0)}$$

"interaction-picture time ~~ev~~ evolution operator"

Initial condition: $U(t_0,t_0) = 1$

Differentiate $U(t,t_0)$ wrt t:

$$\frac{\partial}{\partial t} U(t,t_0) = e^{iH_0(t-t_0)}\; iH_0\; e^{-iH(t-t_0)} + e^{iH_0(t-t_0)}\; (-iH)\; e^{-iH(t-t_0)}$$

$$= -i\; e^{iH_0(t-t_0)}\; \underbrace{(H-H_0)}_{H_{int}}\; e^{-iH(t-t_0)}$$

$$\Rightarrow: \underline{i\frac{\partial}{\partial t} U(t,t_0)} = e^{iH_0(t-t_0)}\; H_{int}\; e^{-iH(t-t_0)}$$

$$= e^{iH_0(t-t_0)}\; H_{int}\; \underbrace{e^{-iH_0(t-t_0)}\, e^{iH_0(t-t_0)}}_{I}\; e^{-iH(t-t_0)}$$

$$= \underline{H_I(t)\; U(t,t_0)}$$

where

$$H_I(t) \equiv e^{iH_0(t-t_0)}\; H_{int}\; e^{-iH_0(t-t_0)}$$

Note that

$$H_I(t) = e^{iH_0(t-t_0)} H_{int} e^{-iH_0(t-t_0)}$$

$$= e^{iH_0(t-t_0)} \int d^3x \frac{\lambda}{4!} \varphi^4(\vec{x},t_0) e^{-iH_0(t-t_0)}$$

$$= \int d^3x \frac{\lambda}{4!} \quad e^{iH_0(t-t_0)} \varphi(\vec{x},t_0) e^{-iH_0(t-t_0)}$$
$$e^{iH_0(t-t_0)} \varphi(\vec{x},t_0) e^{-iH_0(t-t_0)}$$
$$e^{iH_0(t-t_0)} \varphi(\vec{x},t_0) e^{-iH_0(t-t_0)}$$
$$e^{iH_0(t-t_0)} \varphi(\vec{x},t_0) e^{-iH_0(t-t_0)}$$

(where we simply inserted $I = e^{-iH_0(t-t_0)} e^{iH_0(t-t_0)}$ between adjacent φ's)

$$= \int d^3x \frac{\lambda}{4!} \varphi_I^4(\vec{x},t)$$

Thus $H_I(t)$ is given in terms of $\varphi_I(\vec{x},t)$ which has a simple free-field time evolution

The solution of $\quad i \frac{\partial}{\partial t} U(t,t_0) = H_I(t)\, U(t,t_0)$
with $U(t_0,t_0) = 1$ is

$$U(t,t_0) = 1 + (-i) \int_{t_0}^{t} dt_1\, H_I(t_1)$$
$$+ (-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_I(t_1)\, H_I(t_2)$$
$$+ \ldots$$

Proof: differentiate the solution $\Rightarrow$

$$\frac{\partial}{\partial t} U(t,t_0) = 0 + (-i)\, H_I(t)$$
$$+ (-i)^2 \int_{t_0}^{t} dt_2\, H_I(t)\, H_I(t_2) + \ldots$$

$$\Rightarrow i \frac{\partial}{\partial t} U(t,t_0) = H_I(t) \left[1 + (-i) \int_{t_0}^{t_2} dt_2\, H_I(t_2) + \ldots \right]$$
$$= H_I(t)\, U(t,t_0) \qquad \text{OK}$$

(and clearly the solution also satisfies $U(t_0,t_0) = 1$)

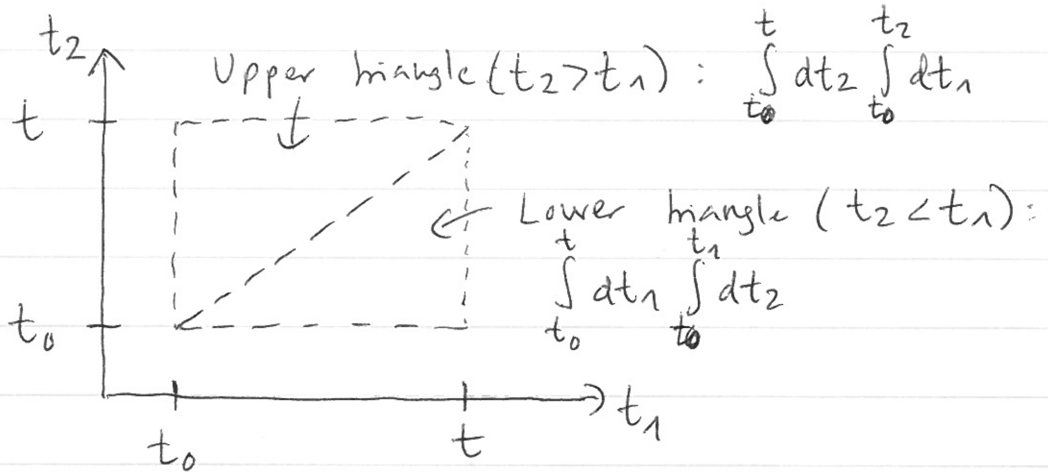
Next note that in the given solution, in each term the H_I factors stand in time order (i.e. later times to the left)
One can rewrite the term containing n factors of H_I as

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \; H_I(t_1) \cdots H_I(t_n)$$

$$= \frac{1}{n!} \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n \; T\{H_I(t_1) H_I(t_2) \cdots H_I(t_n)\}$$

(note that on the right-hand side the upper limit of all integrals is t)

We won't prove this for general n, but verify it holds for $n=2$:



$$\frac{1}{2!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \; T\{H_I(t_1) H_I(t_2)\}$$

(split integral into upper and lower triangle)

$$= \frac{1}{2!} \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \; H_I(t_2) H_I(t_1) \quad \text{(upper: let } t_1 \leftrightarrow t_2\text{)}$$

$$+ \frac{1}{2!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \; H_I(t_1) H_I(t_2) \quad \text{(lower)}$$

$$= \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \quad H_I(t_1)\, H_I(t_2) \qquad \text{OK}$$

Thus we have

$$U(t,t_0) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n \; T\{H_I(t_1) \cdots H_I(t_n)\}$$

This can be written

$$U(t,t_0) = T\left\{ \exp\left[-i \int_{t_0}^{t} dt'\, H_I(t')\right]\right\}$$

(To see this, use $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$)

It will be useful to generalize this so that the second argument of U can take values different from the reference time t_0. We define

$$U(t,t') \equiv T\left\{ \exp\left[-i \int_{t'}^{t} dt''\, H_I(t'')\right]\right\}$$

$(t \geq t')$

By the same type of proof as earlier, one can show that $U(t,t')$ satisfies the same differential equation as before, i.e.

$$i \frac{\partial}{\partial t} U(t,t') = H_I(t)\, U(t,t')$$

Furthermore, this def. of $U(t,t')$ clearly satisfies $U(t',t') = 1$ (initial condition)

In the tutorial for week 10 (see Exercise 2) it is shown that $U(t,t')$ can be written

$$U(t,t') = e^{iH_0(t-t_0)}\, e^{-iH(t-t')}\, e^{-iH_0(t'-t_0)},$$

~~is~~ that $U(t,t')$ is a unitary operator, and satisfies (here $t_1 \geq t_2 \geq t_3$)

(*)
$$U(t_1,t_2)\, U(t_2,t_3) = U(t_1,t_3)$$
$$U(t_1,t_3)\, U^\dagger(t_2,t_3) = U(t_1,t_2)$$

We will make use of these relations later.

Next, we want to find an expression for the ground state $|\Omega\rangle$ of H.

Notation: $|\Omega_n\rangle$, E_n are eigenstates and eigenvalues of H (let $\overset{\text{label}}{n} = 0,1,2,\ldots$ for simplicity). Ground state of H: $|\Omega\rangle \equiv |\Omega_0\rangle$
Ground state energy of H: E_0

Resolution of the identity in terms of $\{|\Omega_n\rangle\}$:

$$I = \sum_n |\Omega_n\rangle\langle\Omega_n|$$

Furthermore, let $|0\rangle$ be the ground state of H_0 with the zero of energy defined s.t. $H_0 |0\rangle = 0$.

This gives

$$e^{-iHt} |0\rangle = e^{-iHt} \overbrace{\sum_n |\Omega_n\rangle\langle\Omega_n|}^{I} 0\rangle$$

$$= \sum_n e^{-iE_n t} |\Omega_n\rangle \langle \Omega_n | 0\rangle$$

We assume that there is a nonzero overlap between the ground states of H and H_0, i.e.

$$\langle \Omega | 0\rangle \; \left(= \langle \Omega_0 | 0 \rangle \right) \neq 0$$

(otherwise $H - H_0 = H_{int}$ could not be considered a small perturbation)

Allow t to be slightly imaginary by defining

$$t = \underset{\substack{\uparrow \\ \text{real}}}{t_{real}} \, (1 - i\underset{\substack{\uparrow \\ \epsilon:\ \text{infinitesimal, real and } > 0}}{\epsilon})$$

Then

$$\frac{e^{-iE_n t}}{e^{-iE_0 t}} = e^{-i(E_n - E_0)\, t_{real}\, (1 - i\epsilon)}$$

$$= e^{-i(E_n - E_0)\, t_{real}} \; \underbrace{e^{-\overbrace{(E_n - E_0)}^{>0} t_{real}\, \epsilon}}_{\longrightarrow\, 0 \text{ as } t_{real} \to \infty}$$

Thus as $t \to \infty(1-i\epsilon)$ the $n \neq 0$ terms in the sum are negligible compared to the $n=0$ term, i.e.

$$e^{-iHt}|0\rangle \overset{t\to\infty(1-i\epsilon)}{\approx} e^{-iE_0t}|\Omega\rangle\langle\Omega|0\rangle$$

$$\Rightarrow |\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)} \left(e^{-iE_0t}\langle\Omega|0\rangle\right)^{-1} e^{-iHt}|0\rangle$$

One next argues that, given the limit here, one can shift t by a finite constant without changing the result. Taking this constant to be our reference time t_0 we get

$$|\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)} \cdot \left(e^{-iE_0(t+t_0)}\langle\Omega|0\rangle\right)^{-1} e^{-iH(t+t_0)}|0\rangle$$

$$= \lim_{t\to\infty(1-i\epsilon)} \left(e^{-iE_0(t_0-(-t))}\langle\Omega|0\rangle\right)^{-1}$$

$$\cdot\, e^{-iH(t_0-(-t))}|0\rangle$$

insert $e^{iH_0(t_0-t_0)}=1$ insert $e^{-iH_0(-t-t_0)}$ and use $H_0|0\rangle=0$ $\Rightarrow e^{-iH_0\alpha}|0\rangle=|0\rangle$

$$U(t_0,-t) = \underbrace{e^{iH_0(t_0-t_0)}e^{-iH(t_0-(-t))}e^{-iH_0(-t-t_0)}}$$

$\Rightarrow$: $$|\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)} \left(e^{-iE_0(t_0-(-t))}\langle\Omega|0\rangle\right)^{-1} \cdot U(t_0,-t)|0\rangle$$

Similarly, it can be shown that

$$\langle\Omega| = \lim_{t\to\infty(1-i\epsilon)} \langle 0|U(t,t_0)\left(e^{-iE_0(t-t_0)}\langle 0|\Omega\rangle\right)^{-1}$$

Now consider the 2-point function:

$$\langle\Omega|T\{\varphi(x)\varphi(y)\}|\Omega\rangle$$

Assume $x^0 > y^0 > t_0$ for now. Since $x^0 > y^0$, T does nothing, giving

$$\langle\Omega|\varphi(x)\varphi(y)|\Omega\rangle$$

$$= \lim_{t\to\infty(1-i\epsilon)} \left(e^{-iE_0(t-t_0)}e^{-iE_0(t_0-(-t))}|\langle\Omega|0\rangle|^2\right)^{-1}$$

$$\langle 0|U(t,t_0)\underbrace{\varphi(x)}_{U^\dagger(x^0,t_0)\varphi_I(x)U(x^0,t_0)}\underbrace{\varphi(y)}_{U^\dagger(y^0,t_0)\varphi_I(y)U(y^0,t_0)}U(t_0,-t)|0\rangle$$

$$= \lim_{t\to\infty(1-i\epsilon)} \left(e^{-2iE_0 t} \, |\langle 0|\Omega\rangle|^2 \right)^{-1}$$

$$\langle 0| \overbrace{U(t,t_0)\, U^\dagger(x^0,t_0)}^{=U(t,x^0)} \varphi_I(x) \overbrace{U(x^0,t_0)\, U^\dagger(y^0,t_0)}^{=U(x^0,y^0)}$$

$$\varphi_I(y) \underbrace{U(y^0,t_0)\; U(t_0,-t)}_{=U(y^0,-t)} |0\rangle$$

when we used the relations $(*)$ on p. 8.

Next, divide this expression by 1 in the form

$$1 = \langle \Omega|\Omega\rangle$$

$$= \lim_{t\to\infty(1-i\epsilon)} \left(e^{-iE_0(t-t_0)} \langle 0|\Omega\rangle \, e^{-iE_0(t_0+t)} \langle\Omega|0\rangle \right)^{-1}$$

$$\cdot \langle 0| \underbrace{U(t,t_0)\; U(t_0,-t)}_{=U(t,-t)} |0\rangle$$

The factor $(\cdots)^{-1}$ cancels, giving

$$\langle\Omega|\varphi(x)\varphi(y)|\Omega\rangle$$

$$= \frac{\langle 0|\, U(t,x^0)\, \varphi_I(x)\, U(x^0,y^0)\, \varphi_I(y)\, U(y^0,-t)|0\rangle}{\langle 0|\, U(t,-t)|0\rangle}$$

Note that all operators are in time order. This would have been the case also for $y^0 > x^0$. The time-ordering operator T can then be introduced to write the time-ordered expressions more compactly, giving our final expression for the two-point function:

$$\langle \Omega | T\{ \varphi(x) \varphi(y) \} | \Omega \rangle$$

$$= \lim_{t \to \infty(1-i\epsilon)} \frac{\langle 0 | T\{ \varphi_I(x) \varphi_I(y) \exp[-i \int_{-t}^{t} dt' H_I(t')] \} | 0 \rangle}{\langle 0 | T\{ \exp[-i \int_{-t}^{t} dt' H_I(t')] \} | 0 \rangle}$$

This expression is the starting point for the perturbation expansion of the two-point function.