Part 2

Starting formula:

$$\langle \Omega | T\{ \varphi(x)\varphi(y) \} | \Omega \rangle$$

$$= \lim_{t \to \infty(1-i\epsilon)}$$

(*)
$$\frac{\langle 0 | T\{ \varphi_I(x)\varphi_I(y) \exp[-i\int_{-t}^{t} dt' H_I(t')] \} | 0 \rangle}{\langle 0 | T\{ \exp[-i\int_{-t}^{t} dt' H_I(t')] \} | 0 \rangle}$$

Since $H_I(t) = \int d^3x \, \frac{\lambda}{4!} \varphi_I^4(\vec{x}, t)$

it follows by writing $\exp[\dots]$ as a series that both the numerator and denominator of (*) are given by sums of expressions of the form

$$\langle 0 | T\{ \varphi_I(x_1)\varphi_I(x_2) \cdots \varphi_I(x_n) \} | 0 \rangle$$

We therefore need to learn how to calculate such vacuum expectation values of time-ordered products of field operators in the interaction picture.

For $n = 2$:

$$\langle 0 | T\{ \varphi_I(x_1)\, \varphi_I(x_2) \} | 0 \rangle = D_F(x_1 - x_2)$$

For $n = 4$:

$$\langle 0 | T\{ \varphi_I(x_1)\, \varphi_I(x_2)\, \varphi_I(x_3)\, \varphi_I(x_4) \} | 0 \rangle$$

$$\begin{aligned} &= D_F(x_1 - x_2)\, D_F(x_3 - x_4) \\ &+ D_F(x_1 - x_3)\, D_F(x_2 - x_4) \\ &+ D_F(x_1 - x_4)\, D_F(x_2 - x_3) \end{aligned}$$

Procedure for general n:

$$\langle 0 | T\{ \varphi_I(x_1) \cdots \varphi_I(x_n) \} | 0 \rangle$$

= sum of products of Feynman propagators D_F. Each term is obtained by pairing $\varphi_I(x_1), \ldots, \varphi_I(x_n)$ in a particular way and replacing each pair $\varphi_I(x_i)\, \varphi_I(x_j)$ by a Feynman propagator $D_F(x_i - x_j)$.

(This result is a consequence of Wick's theorem, which is an operator identity for free fields that we will not ~~p~~ state or prove here.)

So, revisiting the $n=4$ case, we work it out as follows (write $1 = \varphi_I(x_1)$ etc. for short)

$$\langle 0 | T\{1234\} | 0\rangle$$

$$= \langle 0 | T\{1234\} | 0\rangle$$

$$+ \langle 0 | T\{1234\} | 0\rangle$$

$$+ \langle 0 | T\{1234\} | 0\rangle$$

$$= D_F(x_1 - x_2)\, D_F(x_3 - x_4) + D_F(x_1 - x_3)\, D_F(x_2 - x_4)$$
$$+ D_F(x_1 - x_4)\, D_F(x_2 - x_3)$$

Note that for n <u>odd</u>, the answer is 0 because it is impossible to pair up an odd number of objects.

How many terms are there for an <u>even</u> n?

$n-1$ ways to pair up the first field
$n-3$ ways to pair up the next (unpaired) field
$n-5$ . . . . —

$\Rightarrow$: $1 \cdot 3 \cdot 5 \ldots (n-1) \equiv (n-1)!!$

* called "double factorial" (potentially confusing notation since $n!! \neq (n!)!$

$n=2$: $1!! = 1$
$n=4$: $3!! = 3$
$n=6$: $5!! = 15$
etc.

$\Rightarrow$ The number of terms grow very rapidly with n. But some of the terms will be equal if some arguments x_i are equal