Part 4

We have

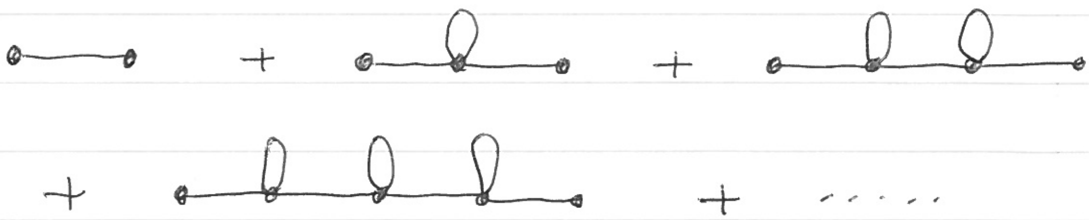
$$\langle \Omega | T \{ \varphi(x) \varphi(y) \} | \Omega \rangle$$

= sum of all connected diagrams

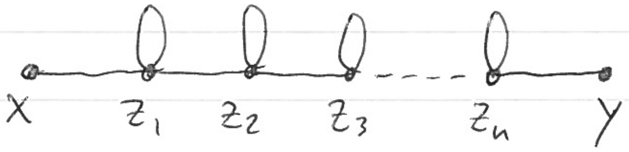
= 

There is an infinite number of diagrams. While summing all of them is not possible, we can sum infinite subsets.

Example:



In this sum, the $O(\lambda^n)$ contribution is



(n bubbles)

$$= 2^{-n} (-i\lambda)^n \int dz_1 \cdots \int dz_n$$

$$D_F(x - z_1)\, D_F(z_1 - z_1)\, D_F(z_1 - z_2)\, D_F(z_2 - z_2)$$

$$\cdots\; D_F(z_{n-1} - z_n)\, D_F(z_n - z_n)\, D_F(z_n - y)$$

$$= \left(- \frac{i\lambda D_F(0)}{2} \right)^n \int dz_1 \dots \int dz_n$$

$$D_F(x-z_1)\, D_F(z_1-z_2) \cdots D_F(z_{n-1}-z_n)\, D_F(z_n-y)$$

$$= \left(- \frac{i\lambda D_F(0)}{2} \right)^n \int dz_1 \dots \int dz_n$$

$$\cdot \int \frac{d^4 p}{(2\pi)^4} \frac{i e^{-ip(x-z_1)}}{p^2 - m^2 + i\epsilon}$$

$$\cdot \int \frac{d^4 p_1}{(2\pi)^4} \frac{i e^{-ip_1(z_1-z_2)}}{p_1^2 - m^2 + i\epsilon}$$

$$\vdots$$

$$\cdot \int \frac{d^4 p_{n-1}}{(2\pi)^4} \frac{i e^{-ip_{n-1}(z_{n-1}-z_n)}}{p_{n-1}^2 - m^2 + i\epsilon}$$

$$\cdot \int \frac{d^4 p_n}{(2\pi)^4} \frac{i e^{-ip_n(z_n-y)}}{p_n^2 - m^2 + i\epsilon}$$

Do the integrals successively in the following order:

$z_n,\ p_n,\ z_{n-1},\ p_{n-1},\ \dots,\ z_1,\ p_1,$

leaving at the end the integral over p.

Integrate over $z_n \Rightarrow (2\pi)^4 \delta(p_n - p_{n-1})$
Integrate over $p_n \Rightarrow$

$$\left(-\frac{i\lambda D_F(0)}{2}\right)^n \int dz_1 \cdots \int dz_{n-1}$$

$$\int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip(x-z_1)}}{p^2 - m^2 + i\epsilon}$$

$$\vdots$$

$$\int \frac{d^4p_{n-2}}{(2\pi)^4} \frac{i e^{-ip_{n-2}(z_{n-2} - z_{n-1})}}{p_{n-2}^2 - m^2 + i\epsilon}$$

$$\cdot \int \frac{d^4p_{n-1}}{(2\pi)^4} \frac{i^2 e^{-ip_{n-1}(z_{n-1} - y)}}{(p_{n-1}^2 - m^2 + i\epsilon)^2}$$

Continue like this ($n-1$ more z_i, p_i integr.)

$$\Rightarrow \left(-\frac{i\lambda D_F(0)}{2}\right)^n \int \frac{d^4p}{(2\pi)^4} \left(\frac{i}{p^2 - m^2 + i\epsilon}\right)^{n+1} e^{-ip(x-y)}$$

So the sum of this infinite subset of diagrams becomes

$$\int \frac{d^4p}{(2\pi)^4} \underbrace{\left[\sum_{n=0}^{\infty} \left(\frac{-i\lambda D_F(0)}{2}\right)^n \left(\frac{i}{p^2 - m^2 + i\epsilon}\right)^{n+1}\right]}_{\text{geometric series}} e^{-ip(x-y)}$$

The geometric series factor is

$$\frac{i}{p^2-m^2+i\epsilon} \sum_{n=0}^{\infty} \left(\frac{\lambda D_F(0)/2}{p^2-m^2+i\epsilon} \right)^n$$

$$= \frac{i}{p^2-m^2+i\epsilon} \cdot \frac{1}{1 - \frac{\lambda D_F(0)}{2} \cdot \frac{1}{p^2-m^2+i\epsilon}}$$

$$= \frac{i}{p^2 - \left(m^2 + \frac{\lambda}{2} D_F(0) \right) + i\epsilon}$$

Let us denote the two-point function as

$$\langle \Omega | T\{ \varphi(x) \varphi(y) \} | \Omega \rangle \equiv D_F(x-y)_{int}$$

$$\equiv \int \frac{d^4 p}{(2\pi)^4} \tilde{D}_F(p)_{int} \, e^{-ip(x-y)}$$

Thus the infinite sum

gives the following approximate result for $\tilde{D}_F(p)_{int}$:

$$\tilde{D}_F(p)_{\text{int}} \simeq \frac{i}{p^2 - \left(m^2 + \frac{\lambda}{2} D_F(0)\right) + i\epsilon} \qquad (*)$$

Recall that for the noninteracting (free) theory (i.e. $\lambda = 0$) we found earlier

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \tilde{D}_F(p)\, e^{-ip(x-y)}$$

with $\tilde{D}_F(p) = \frac{i}{p^2 - m^2 + i\epsilon}$

The dispersion relation $E_{\vec{p}}^2 = \vec{p}^{\,2} + m^2$ was obtained from the poles of $\tilde{D}_F(p)$. Let us now do the same for the interacting theory. From the poles of $(*)$ we then get

$$E_{\vec{p}}^2 = \vec{p}^{\,2} + \left(m^2 + \frac{\lambda}{2} D_F(0)\right)$$

Due to the interactions, the poles have moved, changing the dispersion relation. We interpret the term $\left(m^2 + \frac{\lambda}{2} D_F(0)\right)$ as m_{phys}^2, where m_{phys} is the physical mass of the particles. Thus m_{phys} is different from the mass parameter m in the Lagrangian, which therefore often is renamed as m_{bare} (or m_0)

Thus

$$m^2_{phys} = m^2_{bare} + \frac{\lambda}{2} D_F(0)$$

up to corrections of $O(\lambda^2)$. Note that m_{bare} is not experimentally measurable, only the physical mass m_{phys} is. We say that quantum fluctuations, generated by the interactions, have "dressed" the "bare" quantity m_{bare}, turning it into the physical ("dressed") quantity m_{phys}.

Let us now look at the "mass correction" term $\frac{\lambda}{2} D_F(0)$:

$$D_F(0) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon}$$

this is m_{bare}, but I continue to write m for simplicity

see JOA, Eq. (5.73)

$$\simeq \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\sqrt{\vec{p}^2 + m^2}}$$

$$= \frac{1}{(2\pi)^3} \frac{1}{2} \int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos\theta) \int_0^\infty dp\, p^2 \frac{1}{\sqrt{p^2 + m^2}}$$

$$= \frac{1}{4\pi^2} \int_0^\infty dp \frac{p^2}{\sqrt{p^2 + m^2}}$$

The integral diverges at large p ("ultraviolet divergence"). But we have no reason to trust the theory at arbitrarily high momentum (energy) scales. Let us assume the theory is valid up to some momentum scale $\Lambda \gg m$. We regularize the integral by cutting it off at $p = \Lambda$. This gives our new, regularized version of $D_F(0)$:

$$D_F(0) = \frac{1}{4\pi^2} \int_0^\Lambda dp \frac{p^2}{\sqrt{p^2+m^2}}$$

$$= \frac{1}{8\pi^2} \left[\Lambda \sqrt{\Lambda^2+m^2} - m^2 \log \frac{\Lambda + \sqrt{\Lambda^2+m^2}}{m} \right]$$

$$\overset{\Lambda \gg m}{=} \frac{1}{8\pi^2} \left[\Lambda^2 - \frac{1}{2} m^2 \left(\log \frac{\Lambda^2}{m^2} - c \right) \right]$$

when c is an $O(1)$ constant. This gives

$$m_{phys}^2 = m_{bare}^2 + \frac{\lambda}{16\pi^2} \Big[\Lambda^2 - \frac{1}{2} m_{bare}^2 \left(\log \frac{\Lambda^2}{m_{bare}^2} - c \right) \Big]$$

Note that this relation between m_{phys} and m_{bare} depends on Λ. Since m_{phys} (which we can measure) cannot depend on our choice of Λ (which contains arbitrariness), the bare mass m_{bare} must depend on Λ in such a way that m_{phys} does not. Thus if we change Λ

we must also adjust m_{bare} to keep m_{phys} the same.

In general, all quantities in the Lagrangian density will depend on the cutoff Λ. For example, consider the Lagrangian density of quantum electrodynamics (QED), which describes the Dirac and EM fields and their coupling:

$$\mathcal{L} = \bar{\Psi}\left[\gamma^\mu (i\partial_\mu - e_{bare} A_\mu) - m_{bare}\right]\Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Here the bare charge and mass parameters e_{bare} and m_{bare} both depend on the cutoff Λ. ~~The~~ $e_{bare}(\Lambda)$ and $m_{bare}(\Lambda)$ are chosen s.t. theory and experiment agree for <u>two</u> chosen physical processes/quantities. Once one has determined $e_{bare}(\Lambda)$ and $m_{bare}(\Lambda)$ in this way, all other predictions of the theory (pertaining to physics on energy scales $\ll \Lambda$) are found to agree with experiment to very high accuracy.

Attempting a brief definition, one can say that the theory behind, and methods involved, in describing/analyzing the relationship between bare and dressed (physical) quantities, and how the ~~former~~ depend on the cutoff Λ, are referred to as <u>renormalization</u>.