Momentum-space Feynman diagrams and Feynman rules (lectured Thursday 22 March)

Earlier we have looked at the perturbation expansion for the two-point function $\langle \Omega | T\{\varphi(x)\varphi(y)\} | \Omega \rangle \equiv D_F(x-y)_{\text{int}}$ in φ^4 theory. We discussed a diagrammatic representation of the terms in the expansion. The diagrams we looked at are known as position-space Feynman diagrams, as they involve space-time points (x and y) as variables. Often it is however more convenient to consider the Fourier transformation $\widetilde{D}_F(p)_{\text{int}}$ of $D_F(x-y)_{\text{int}}$, defined via

$$D_F(x-y)_{\text{int}} = \int \frac{d^4p}{(2\pi)^4} \widetilde{D}_F(p)_{\text{int}}\, e^{-ip(x-y)}$$

In other words, one expresses $\widetilde{D}_F(p)_{\text{int}}$ (rather than $D_F(x-y)_{\text{int}}$) as a sum of Feynman diagrams. These diagrams are called momentum-space Feynman diagrams (since they depend on the momentum variable p) and the corresponding Feynman rules are called momentum-space Feynman rules.

As a concrete example, consider the position-space Feynman diagram

which gives the following contribution to $D_F(x-y)_{int}$ (found by using the position-space Feynman rules):

$$\frac{1}{S}(-i\lambda)\int d^4z\, D_F(x-z)\, D_F(z-z)\, D_F(z-y)$$

$$= \frac{1}{S}(-i\lambda)\int d^4z\, D_F(x-z)\, D_F(0)\, D_F(z-y)$$

(where $S=2$ for this diagram; S is the symmetry factor)

The corresponding contribution to $\tilde{D}_F(p)_{int}$ is found by writing

$$\tilde{D}_F(p)_{int} = \int d^4(x-y)\, D_F(x-y)_{int}\, e^{ip(x-y)}$$

$$= \int d^4x\, D_F(x)_{int}\, e^{ipx}$$

and inserting the contribution to $D_F(x)_{int}$ from the above diagram. This gives

$$\text{contribution to } \tilde{D}_F(p)_{int} = \int d^4x\, \frac{1}{S}(-i\lambda)$$

$$\int d^4z\, D_F(x-z)\, D_F(0)\, D_F(z)\, e^{ipx}$$

$$= \frac{1}{S}(-i\lambda) \int d^4x \int d^4z$$

$$\int \frac{d^4p_1}{(2\pi)^4} \frac{i}{p_1^2 - m^2 + i\epsilon} e^{-ip_1(x-z)}$$

$$\int \frac{d^4p_2}{(2\pi)^4} \frac{i}{p_2^2 - m^2 + i\epsilon} \cdot \int \frac{d^4p_3}{(2\pi)^4} \frac{i}{p_3^2 - m^2 + i\epsilon} e^{-ip_3z} e^{ipx}$$

Do $\int d^4x \Rightarrow (2\pi)^4 \delta(p - p_1)$

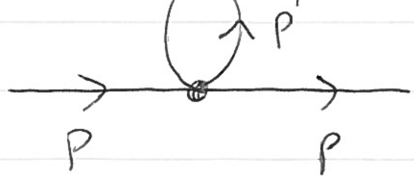
Do $\int \frac{d^4p_1}{(2\pi)^4} \Rightarrow \ldots$

Do $\int d^4z \Rightarrow (2\pi)^4 \delta(p - p_3)$

Do $\int \frac{d^4p_3}{(2\pi)^4} \Rightarrow \ldots$ (and rename p_2 as p')

$$= \frac{1}{S}(-i\lambda) \frac{i}{p^2 - m^2 + i\epsilon} \left(\int \frac{d^4p'}{(2\pi)^4} \frac{i}{p'^2 - m^2 + i\epsilon} \right) \frac{i}{p^2 - m^2 + i\epsilon}$$

This contribution to $\tilde{D}_F(p)_{\text{int}}$ is associated with the <u>momentum</u>-space Feynman diagram

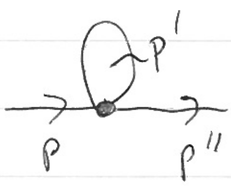


Momentum-space Feynman rules :

* For each propagator , $\xrightarrow[p]{} = \dfrac{i}{p^2 - m^2 + i\epsilon}$
* For each vertex ,  $= -i\lambda$
* Impose momentum conservation at each vertex
* Integrate over each undetermined momentum p' : $\int \dfrac{d^4 p'}{(2\pi)^4}$
* Divide by the symmetry factor S

Example: Check that by using these rules we get the right result for the momentum-space Feynman diagram

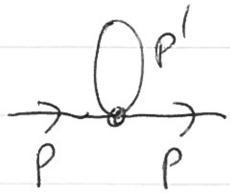
Label each $\rightarrow$ by a momentum variable, i.e.



Momentum conservation at the vertex gives

$$p + p' = p' + p'' \Rightarrow p = p''$$

So the diagram becomes

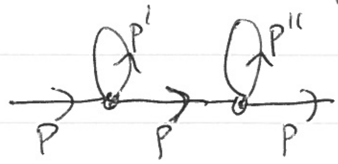


p' is not fixed by the momentum conservation so we must integrate over it. Using the momentum-space Feynman rules then gives

$$\frac{-i\lambda}{2}\,\frac{i}{p^2-m^2+i\epsilon}\left(\int\frac{d^4p'}{(2\pi)^4}\,\frac{i}{p'^2-m^2+i\epsilon}\right)\frac{i}{p^2-m^2+i\epsilon}$$

which agrees with the expression on p. 3, as expected. $(S=2)$

Another example: Find the expression for



(Momentum conservation at the two vertices immediately gives that the momenta can be labeled as shown). p' and p'' are not determined by mom. cons. so need to be integrated over

$$\Rightarrow \left(\frac{i}{p^2-m^2+i\epsilon}\right)^3\frac{(-i\lambda)^2}{S}\int\frac{d^4p'}{(2\pi)^4}\,\frac{i}{p'^2-m^2+i\epsilon}\int\frac{d^4p''}{(2\pi)^4}\,\frac{i}{p''^2-m^2+i\epsilon}$$

where the symmetry factor $S=2^2=4$ for this diagram.