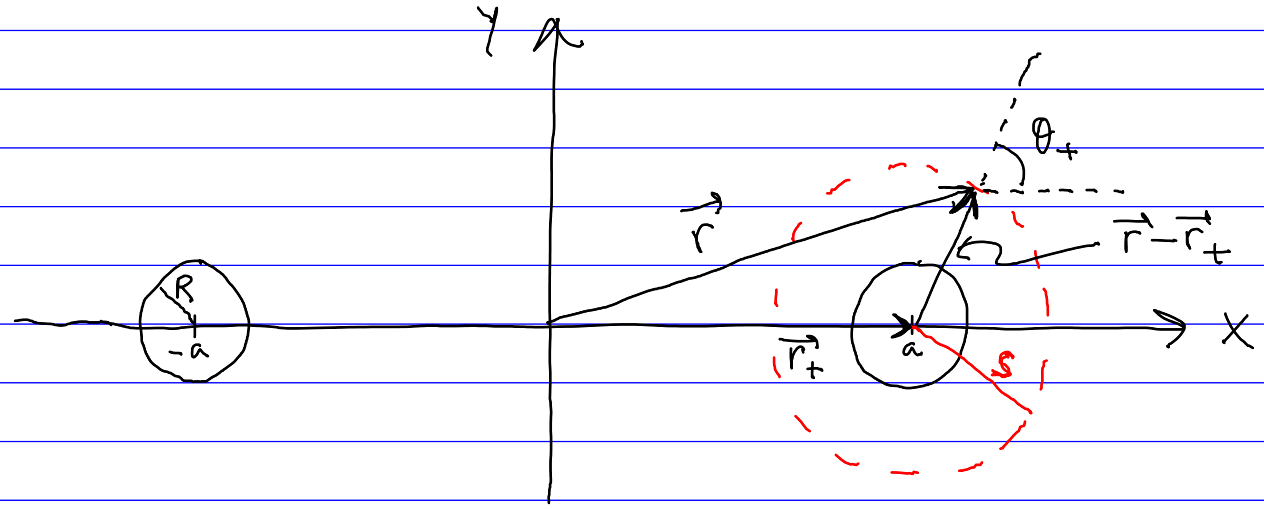
Example: Maxwell stress tensor for two charged cylinders

Two cylinders of radius R are positioned as shown in the figure below. Both extend indefinitely along the z axis. We assume that each cylinder has a <u>uniform</u> charge density on the cylinder surface. Let the surface charge density be σ_+ (σ_-) on the "+" ("−") cylinder centered at $x = +a$ ($x = -a$).



The electric field $\vec{E}_\pm$ produced by the $\pm$-cylinder can, due to its high symmetry, most easily be found from Gauss's law. Picking the Gaussian surface to be a cylinder of length ℓ and radius s ($> R$) centered at the cylinder, there is no contribution from the two "end faces" (with normal vector $\hat{n} = \pm \hat{z}$) since $E_{\pm z} = 0$ by symmetry. Symmetry also dictates that $\vec{E}_\pm$ points radially from each cylinder center and that its magnitude only depends on r. Gauss's law thus gives (where $E_\pm$ is the radial component)

$$\oint_S \vec{E}_\pm \cdot d\vec{a} = \frac{Q_{inside}}{\epsilon_0} \overset{s>R}{\Longrightarrow} E_\pm \cdot \cancel{2\pi} s \cancel{\ell} = \frac{\sigma_\pm \cdot \cancel{2\pi} R \cancel{\ell}}{\epsilon_0}$$

$$\Rightarrow \quad E_\pm = \frac{\sigma_\pm}{\epsilon_0}\frac{R}{s} = \frac{\sigma_\pm R}{\epsilon_0\sqrt{(x\mp a)^2+y^2}}$$

(same sign as $\sigma_\pm$)
(valid for $s > R$)

Let $\vec{R}_\pm \equiv \vec{r} - \vec{r}_\pm$, where $\vec{r}_\pm = (\pm a, 0)$ is the position vector of the $\pm$ cylinder. Then

$$E_{\pm,x} = E_\pm \cos\theta_\pm, \quad E_{\pm,y} = E_\pm \sin\theta_\pm$$

where $\cos\theta_\pm = \hat{R}_\pm \cdot \hat{x} = \dfrac{\vec{r}-\vec{r}_\pm}{|\vec{r}-\vec{r}_\pm|}\cdot\hat{x} = \dfrac{x\mp a}{\sqrt{(x\mp a)^2+y^2}}$

and $\sin\theta_\pm = \dfrac{y}{\sqrt{(x\mp a)^2+y^2}}$

$$\left.\begin{aligned} \Rightarrow \quad E_{\pm,x} &= \frac{\sigma_\pm R}{\epsilon_0}\frac{x\mp a}{(x\mp a)^2+y^2} \\ E_{\pm,y} &= \frac{\sigma_\pm R}{\epsilon_0}\frac{y}{(x\mp a)^2+y^2} \end{aligned}\right\}$$

valid for regions <u>outside</u> $\pm$ cylinder, respectively (inside, $E_\pm = 0$)

The components of the total electric field $\vec{E} = \vec{E}_+ + \vec{E}_-$ are then

$$E_x = E_{+,x} + E_{-,x}$$
$$E_y = E_{+,y} + E_{-,y}$$

The total force on a cylinder can be calculated from

$$\vec{F} = \oint_{\partial\Omega} \overleftrightarrow{T}\cdot d\vec{a}$$

where $\overleftrightarrow{T}$ is Maxwell's stress tensor for the problem and $\partial\Omega$ is a surface enclosing the volume Ω which includes the cylinder of interest (and none of the other cylinder). Since $\vec{E}$ is in the xy plane only F_x and F_y need to be considered. (In fact, $F_y = 0$ by symmetry, which we will also verify explicitly.) So let us consider

$$F_x = \oint_{\partial\Omega} \left(T_{xx}\, da_x + T_{xy}\, da_y + T_{xz}\, da_z \right)$$

$$F_y = \oint_{\partial\Omega} \left(T_{yx}\, da_x + T_{yy}\, da_y + T_{yz}\, da_z \right)$$

We have, for this problem,

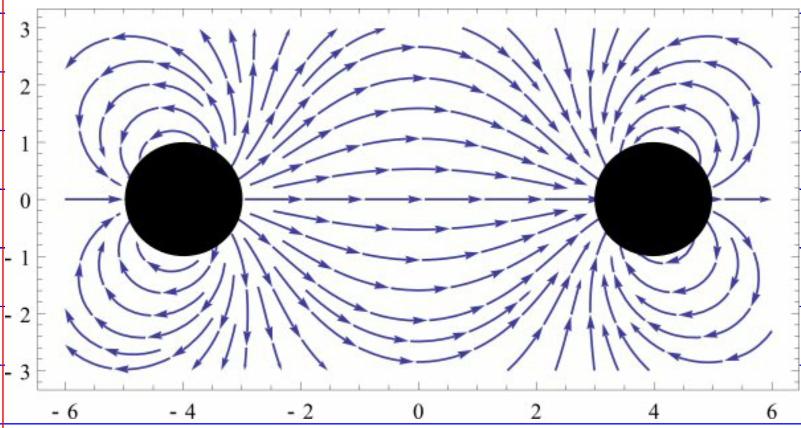
$$T_{xx} = -T_{yy} = \frac{\epsilon_0}{2} \left(E_x^2 - E_y^2 \right)$$

$$T_{xy} = T_{yx} = \epsilon_0 E_x E_y$$

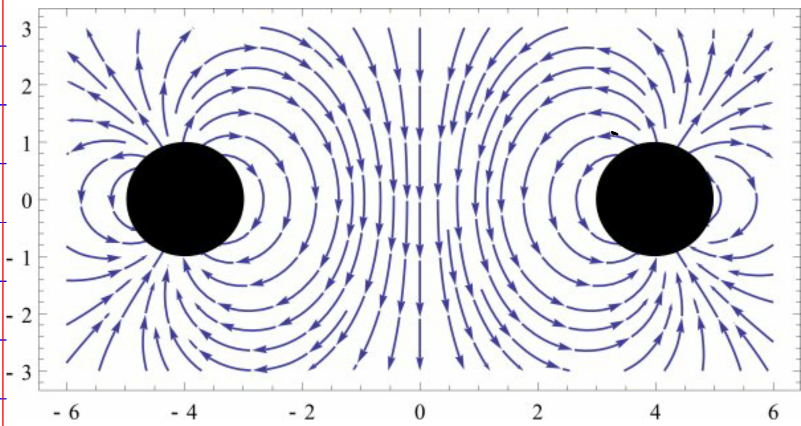
$$T_{yz} = T_{xz} = 0$$

Vector field plots of (T_{xx}, T_{xy}) (relevant for the calculation of F_x) and (T_{yx}, T_{yy}) (relevant for the calculation of F_y) are shown below for the two cases $\sigma_- = -\sigma_+$ (cylinders with opposite charges) and $\sigma_- = \sigma_+$ (—— " —— same charges). (Note that only the relative sign of the charges matter for $\overleftrightarrow{T}$, not which cylinder has which sign of the charge, since $\overleftrightarrow{T}$ is invariant under $\vec{E} \to -\vec{E}$).

Opposite charges

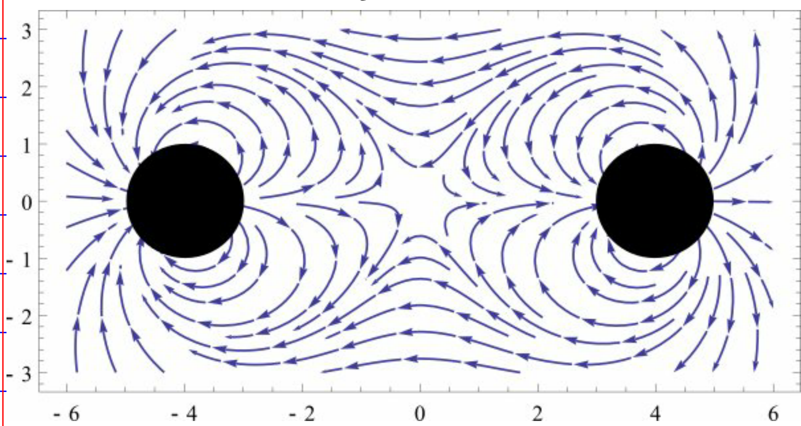


(T_{xx}, T_{xy})

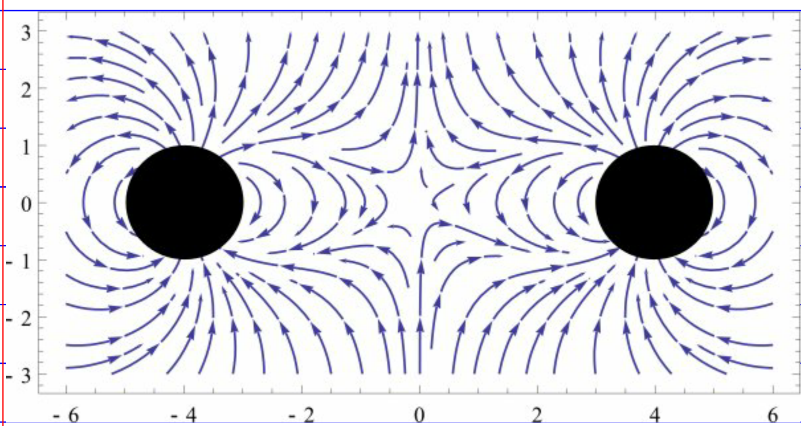


(T_{yx}, T_{yy})

Same charges



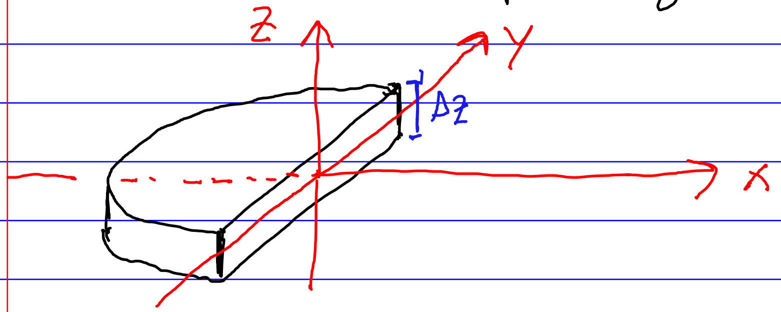
(T_{xx}, T_{xy})



(T_{yx}, T_{yy})

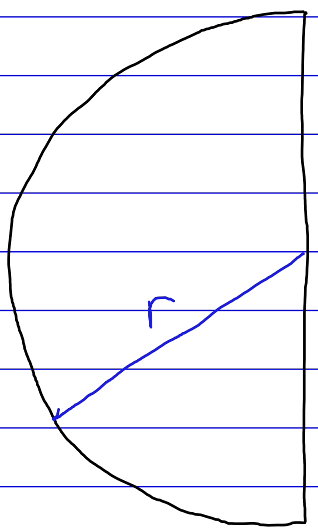
For concreteness let us calculate F_x and F_y on the left ("−") cylinder. The volume Ω can be any volume enclosing this cylinder and none of the other. For symmetry reasons a good choice is to pick Ω to be the region $x < 0$, i.e. the left "half-space".

Consider the following finite volume:



The base area is shaped like a semi-circle with radius r. The height is Δz

view from above



For $r \gg R, a$, the electric field on the circular part of the boundary decays with r as $1/r$ (or faster, if the two cylinders have $\sigma_+ = -\sigma_-$). For the nonzero T_{ij} coefficients, $T_{ij} \sim E^2 \sim (1/r)^2 = 1/r^2$, so the contribution to $\int \overleftrightarrow{T} \cdot d\vec{a}$ from the circular part of the boundary scales like

$$(1/r^2) \cdot \frac{2\pi r}{2} \cdot \Delta z \propto \frac{1}{r}, \quad \text{which} \to 0 \text{ as } r \to \infty$$

Also, the contribution from the top and bottom surfaces vanish since $T_{xz} = T_{yz} = 0$. Thus in the limit $r \to \infty$ the only contribution to $\int \overleftrightarrow{T} \cdot d\vec{a}$ comes from the boundary at $x = 0$. Letting $\Delta z \to 0$ this becomes the plane $x = 0$ (yz plane).

For F_x we get

$$F_x = \int_{\substack{x=0 \\ \text{plane}}} (T_{xx}, T_{xy}) \cdot \underbrace{(da_x, da_y)}_{\substack{= (da_x, 0) \quad \text{since } \hat{n} = \hat{e}_x \\ (\hat{n} \text{ points out of the volume } x<0)}}$$

$$= \int_{-\infty}^{\infty} T_{xx}(x=0,y)\, dy\, L \quad \text{where } L \text{ is the cylinder length}$$

Since $L = \infty$, F_x obviously diverges. Let us therefore calculate the force per unit length given by

$$\left(\lim_{L\to\infty}\right) \frac{F_x}{L} = \int_{-\infty}^{\infty} dy\, T_{xx}(x=0,y)$$

$$= \frac{\epsilon_0}{2} \int_{-\infty}^{\infty} dy \left[E_x^2(x=0,y) - E_y^2(x=0,y) \right]$$

$$= \frac{\epsilon_0}{2} \left(\frac{R}{\epsilon_0}\right)^2 \int_{-\infty}^{\infty} dy \frac{1}{(a^2+y^2)^2} \left\{ \left(\sigma_+(-a) + \sigma_- a\right)^2 - \left(\sigma_+ y + \sigma_- y\right)^2 \right\}$$

$$= \frac{R^2}{2\epsilon_0} \int_{-\infty}^{\infty} dy \frac{1}{(a^2+y^2)^2} \left\{ a^2(\sigma_+ - \sigma_-)^2 - y^2(\sigma_+ + \sigma_-)^2 \right\}$$

$$= \frac{R^2}{2\epsilon_0} \left\{ \frac{\pi}{2} \frac{1}{a^3} \cdot a^2 (\sigma_+ - \sigma_-)^2 - \frac{\pi}{2a} (\sigma_+ + \sigma_-)^2 \right\}$$

looking up the integrals

$$= \frac{\pi R^2}{4\epsilon_0 a}\left\{(\sigma_+ - \sigma_-)^2 - (\sigma_+ + \sigma_-)^2\right\}$$

For opposite charges ($\sigma_+ = -\sigma_- \equiv \sigma$) this gives

$$\underline{\underline{\frac{F_x}{L} = \frac{\pi R^2 \sigma^2}{\epsilon_0 a}}} \qquad (>0\text{, so attracted by right cyl.})$$

while for same charges ($\sigma_+ = \sigma_- \equiv \sigma$)

$$\underline{\underline{\frac{F_x}{L} = -\frac{\pi R^2 \sigma^2}{\epsilon_0 a}}} \qquad (<0\text{, so repelled by right cyl.})$$

Thus as expected, F_x has the same magnitude in both cases but opposite sign. (This result can be shown to be identical to the force/length experienced between two infinite line charges a distance $2a$ apart, if the charge per unit length is the same as for the cylinders, a reasonable result given that the cylinder charge is assumed from the outset to be uniform.)

Finally,

$$F_y = \int_{x=0} T_{yx}\, da_x = L\,\epsilon_0 \int_{-\infty}^{\infty} dy\, E_x(x=0,y)\, E_y(x=0,y)$$

$$\Rightarrow \underline{\underline{\frac{F_y}{L}}} = \epsilon_0 \left(\frac{R}{\epsilon_0}\right)^2 \int_{-\infty}^{\infty} dy \frac{a}{(a^2+y^2)^2}(-\sigma_+ + \sigma_-)(\sigma_+ + \sigma_y)\, y = \underline{\underline{0}}$$

We see that $F_y = 0$ not only for the two cases of identical and opposite charges, but also more generally due to the integrand being odd in y.