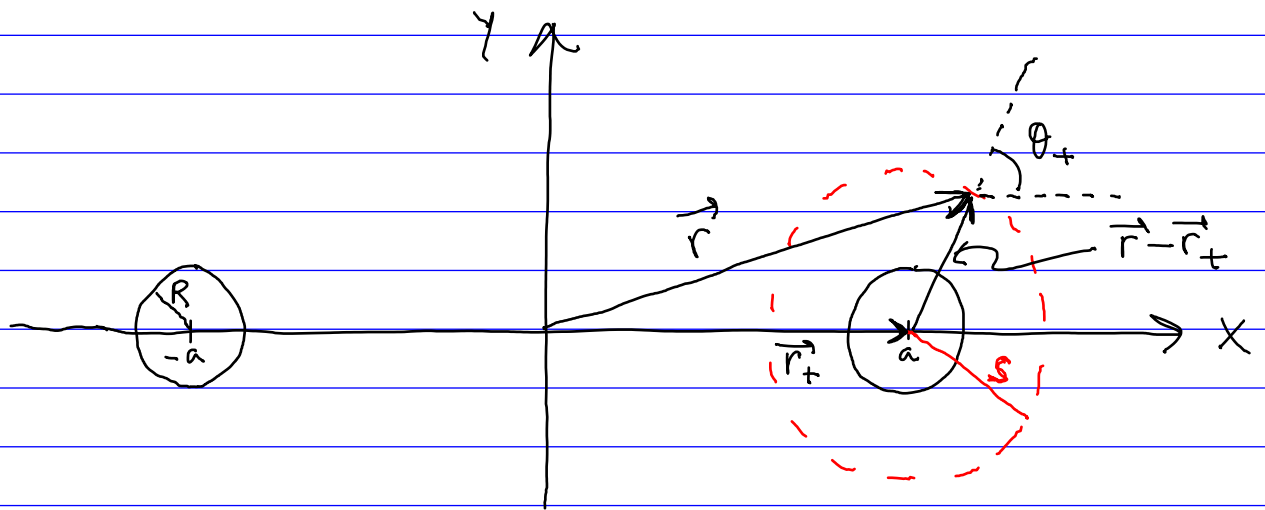


Example : Maxwell stress tensor for two charged cylinders

Two cylinders of radius R are positioned as shown in the figure below. Both extend indefinitely along the z axis. We assume that each cylinder has a uniform charge density on the cylinder surface. Let the surface charge density be σ_+ (σ_-) on the "+" ("−") cylinder centered at $x = +a$ ($x = -a$)



The electric field $\vec{E}_{\pm}$ produced by the $\pm$ -cylinder can, due to its high symmetry, most easily be found from Gauss's law. Picking the Gaussian surface to be a cylinder of length l and radius s ($s > R$) centered at the cylinder, there is no contribution from the two "end faces" (with normal vector $\hat{n} = \pm \hat{z}$) since $E_{\pm z} = 0$ by symmetry. Symmetry also dictates that $\vec{E}_{\pm}$ points radially from each cylinder center and that its magnitude only depends on r . Gauss's law thus gives (where $E_{\pm}$ is the radial component)

$$\oint_S \vec{E}_{\pm} \cdot d\vec{a} = \frac{Q_{\text{inside}}}{\epsilon_0} \quad s > R \Rightarrow E_{\pm} \cdot 2\pi s l = \frac{\sigma_{\pm} \cdot 2\pi R l}{\epsilon_0}$$

$$\Rightarrow E_{\pm} = \frac{\sigma_{\pm} R}{\epsilon_0 s} = \frac{\sigma_{\pm} R}{\epsilon_0 \sqrt{(x \mp a)^2 + y^2}} \quad \begin{matrix} \text{(same sign as } \sigma_{\pm}) \\ \text{(valid for } s > R) \end{matrix}$$

Let $\vec{R}_{\pm} \equiv \vec{r} - \vec{r}_{\pm}$, where $\vec{r}_{\pm} = (\pm a, 0)$ is the position vector of the $\pm$ cylinder. Then

$$E_{\pm, x} = E_{\pm} \cos \theta_{\pm}, \quad E_{\pm, y} = E_{\pm} \sin \theta_{\pm}$$

$$\text{where } \cos \theta_{\pm} = \frac{\hat{R}_{\pm} \cdot \hat{x}}{|\vec{r} - \vec{r}_{\pm}|} = \frac{\vec{r} - \vec{r}_{\pm}}{|\vec{r} - \vec{r}_{\pm}|} \cdot \hat{x} = \frac{x \mp a}{\sqrt{(x \mp a)^2 + y^2}}$$

$$\text{and } \sin \theta_{\pm} = \frac{y}{\sqrt{(x \mp a)^2 + y^2}}$$

$$\Rightarrow \left. \begin{aligned} E_{\pm, x} &= \frac{\sigma_{\pm} R}{\epsilon_0} \frac{x \mp a}{(x \mp a)^2 + y^2} \\ E_{\pm, y} &= \frac{\sigma_{\pm} R}{\epsilon_0} \frac{y}{(x \mp a)^2 + y^2} \end{aligned} \right\} \begin{array}{l} \text{valid for regions} \\ \text{outside } \pm \text{ cylinder,} \\ \text{respectively (inside,} \\ E_{\pm} = 0) \end{array}$$

The components of the total electric field $\vec{E} = \vec{E}_+ + \vec{E}_-$ are then $E_x = E_{+, x} + E_{-, x}$
 $E_y = E_{+, y} + E_{-, y}$

The total force on a cylinder can be calculated from

$$\vec{F} = \oint_{\partial \Omega} \overleftrightarrow{T} \cdot d\vec{a}$$

where $\overleftrightarrow{T}$ is Maxwell's stress tensor for the problem and $\partial \Omega$ is a surface

enclosing the volume Ω which includes the cylinder of interest (and none of the other cylinder). Since $\vec{E}$ is in the xy plane only F_x and F_y need to be considered. (In fact, $F_y = 0$ by symmetry, which we will also verify explicitly.) So let us consider

$$F_x = \oint_{\partial\Omega} (T_{xx} da_x + T_{xy} da_y + T_{xz} da_z)$$

$$F_y = \oint_{\partial\Omega} (T_{yx} da_x + T_{yy} da_y + T_{yz} da_z)$$

We have, for this problem,

$$T_{xx} = -T_{yy} = \frac{\epsilon_0}{2} (E_x^2 - E_y^2)$$

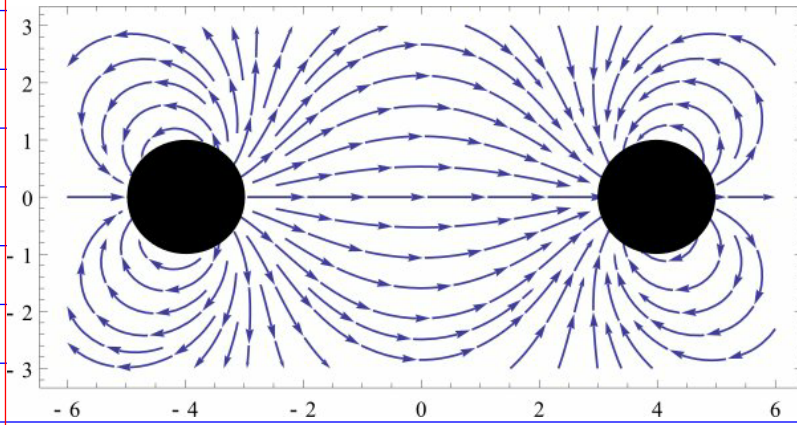
$$T_{xy} = T_{yx} = \epsilon_0 E_x E_y$$

$$T_{yz} = T_{xz} = 0$$

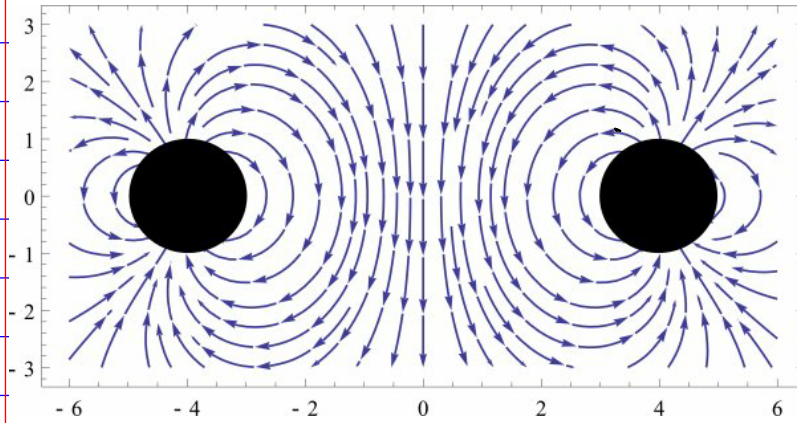
Vector field plots of (T_{xx}, T_{xy}) (relevant for the calculation of F_x) and (T_{yx}, T_{yy}) (relevant for the calculation of F_y) are shown below for the two cases

$\sigma_- = -\sigma_+$ (cylinders with opposite charges) and $\sigma_- = \sigma_+$ (——||—— same charges). (Note that only the relative sign of the charges matter for $\vec{T}$, not which cylinder has which sign of the charge, since $\vec{T}$ is invariant under $\vec{E} \rightarrow -\vec{E}$).

Opposite charges

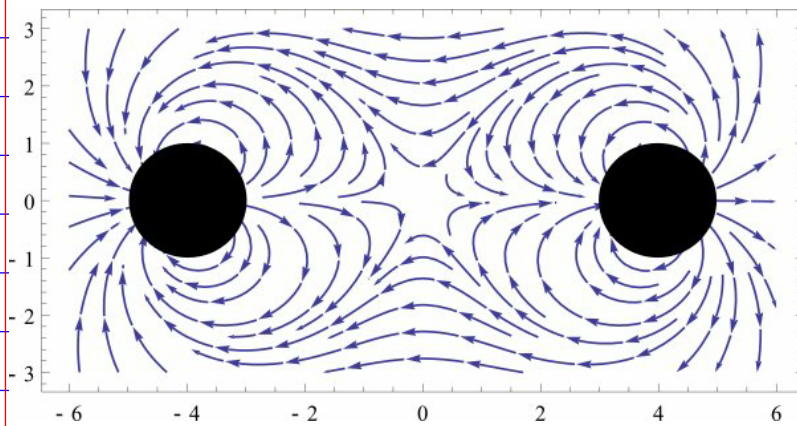


$$(T_{xx}, T_{xy})$$

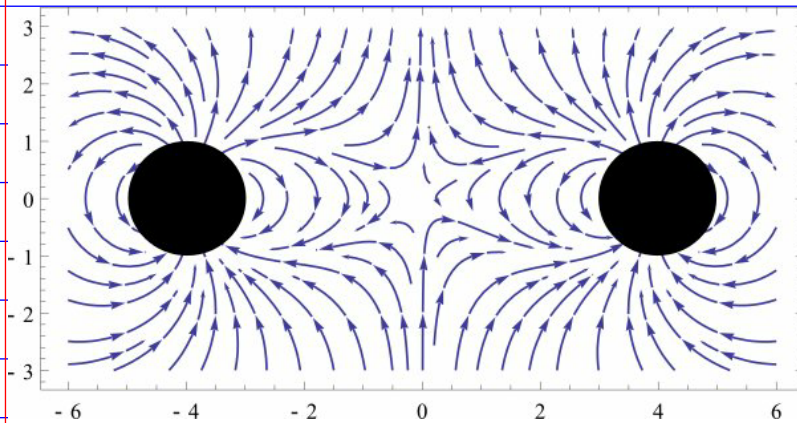


$$(T_{yx}, T_{yy})$$

Same charges



$$(T_{xx}, T_{xy})$$



$$(T_{yx}, T_{yy})$$

For concreteness let us calculate F_x and F_y on the left ("−") cylinder. The volume Ω can be any volume enclosing this cylinder and none of the other. For symmetry reasons a good choice is to pick Ω to be the region $x < 0$, as then we only need to evaluate T_{ij} on the high-symmetry face $x=0$ (the other parts of $\partial\Omega$ are "at infinity" and thus give no contribution since $T_{ij} \rightarrow 0$ there)

For F_x we get

$$F_x = \int_{\substack{x=0 \\ \text{plane}}} (T_{xx}, T_{xy}) \cdot \underbrace{(da_x, da_y)}_{=(da_x, 0) \text{ since } \hat{n} = \hat{e}_x \text{ (}\hat{n} \text{ points out of the volume } x < 0\text{)}}$$

$$= \int_{-\infty}^{\infty} T_{xx}(x=0, y) dy L \quad \text{where } L \text{ is the cylinder length}$$

Since $L = \infty$, F_x obviously diverges. Let us therefore calculate the force per unit length given by

$$\left(\lim_{L \rightarrow \infty} \right) \frac{F_x}{L} = \int_{-\infty}^{\infty} dy T_{xx}(x=0, y)$$

$$= \frac{\epsilon_0}{2} \int_{-\infty}^{\infty} dy \left[E_x^2(x=0, y) - E_y^2(x=0, y) \right]$$

$$\begin{aligned}
&= \frac{\epsilon_0}{2} \left(\frac{R}{\epsilon_0} \right)^2 \int_{-\infty}^{\infty} dy \frac{1}{(a^2 + y^2)^2} \left\{ (\sigma_+(-a) + \sigma_- a)^2 - (\sigma_+ y + \sigma_- y)^2 \right\} \\
&= \frac{R^2}{2\epsilon_0} \int_{-\infty}^{\infty} dy \frac{1}{(a^2 + y^2)^2} \left\{ a^2 (\sigma_+ - \sigma_-)^2 - y^2 (\sigma_+ + \sigma_-)^2 \right\} \\
&= \frac{R^2}{2\epsilon_0} \left\{ \frac{\pi}{2} \frac{1}{a^3} \cdot a^2 (\sigma_+ - \sigma_-)^2 - \frac{\pi}{2a} (\sigma_+ + \sigma_-)^2 \right\} \\
&= \frac{\pi R^2}{4\epsilon_0 a} \left\{ (\sigma_+ - \sigma_-)^2 - (\sigma_+ + \sigma_-)^2 \right\} \quad \leftarrow \text{(looking up the integrals)}
\end{aligned}$$

For opposite charges ($\sigma_+ = -\sigma_- \equiv \sigma$) this gives

$$\underline{\underline{\frac{F_x}{L} = \frac{\pi R^2 \sigma^2}{\epsilon_0 a}}} \quad (> 0, \text{ so attracted by right cyl.})$$

while for same charges ($\sigma_+ = \sigma_- \equiv \sigma$)

$$\underline{\underline{\frac{F_x}{L} = - \frac{\pi R^2 \sigma^2}{\epsilon_0 a}}} \quad (< 0, \text{ so repelled by right cyl.})$$

Thus as expected, F_x has the same magnitude in both cases but opposite sign. (This result can be shown to be identical to the force/length experienced between two infinite line charges a distance $2a$ apart, if the charge per unit length is the same as for the cylinders, a reasonable result given that the cylinder charge is assumed from the outset to be uniform.)

Finally,

$$F_y = \int_{x=0} T_{yx} da_x = L \epsilon_0 \int_{-\infty}^{\infty} dy E_x(x=0, y) E_y(x=0, y)$$

$$\Rightarrow \underline{\underline{\frac{F_y}{L}}} = \epsilon_0 \left(\frac{R}{\epsilon_0} \right)^2 \int_{-\infty}^{\infty} dy \frac{a}{(a^2 + y^2)^2} (-\sigma_+ + \sigma_-)(\sigma_+ + \sigma_-) y$$

$$\underline{\underline{= 0}}$$

We see that $F_y = 0$ not only for the two cases of identical and opposite charges, but also more generally due to the integrand being odd in y .