NTNU – Trondheim
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Department of Physics

Examination paper for
TFY4210 Quantum theory of many-particle systems

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Approved calculator

Rottmann: Matematisk Formelsamling

Rottmann: Matematische Formelsammlung

Barnett & Cronin: Mathematical Formulae

Other information:

The exam has 3 problems. Some formulas can be found on the last page.

The problems were developed by John Ove Fjærestad and discussed with Jens Oluf Andersen.

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Date Signature

Problem 1

Consider the Hamiltonian

$$H = \varepsilon(a_1^\dagger a_1 + a_2^\dagger a_2) + \Delta(a_1^\dagger a_2^\dagger + \text{h.c.}) \tag{1}$$

where $\varepsilon > 0$ and $\Delta > 0$ are parameters, and a_1 and a_2 are fermionic operators satisfying canonical anticommutation relations. Thus $\{a_i, a_j^\dagger\} = \delta_{ij}$ $(i, j = 1, 2)$, and all other anticommutators involving these operators vanish.

In order to write the Hamiltonian in diagonal form we transform to a new set c_1, c_2 of fermionic operators. The transformation reads

$$a_1 = uc_1 - vc_2^\dagger, \tag{2}$$
$$a_2 = uc_2 + vc_1^\dagger, \tag{3}$$

where u and v are real numbers.

(a) Use the requirement that the c-operators should also satisfy canonical anticommutation relations to show that

$$u^2 + v^2 = 1 \tag{4}$$

(to show this, it is sufficient that you calculate just one selected anticommutator). This result can be used to write $u = \cos\theta$, $v = \sin\theta$, where θ is an angle.

(b) Show that by choosing θ such that

$$\tan 2\theta = \frac{\Delta}{\varepsilon} \tag{5}$$

the term proportional to $(c_1^\dagger c_2^\dagger + \text{h.c.})$ in H vanishes.

(c) Show that with this choice, H can be written as

$$H = F(c_1^\dagger c_1 + c_2^\dagger c_2) + G. \tag{6}$$

Give expressions for F and G in terms of ε and Δ (pick the positive solution for $\cos 2\theta$).

(d) Determine the energy and degeneracy of each energy level of H. Here, assume that the only restriction on particle numbers comes from the Pauli principle, i.e. work in the grand canonical ensemble.

Problem 2

Consider a ferromagnet on a square lattice. The Hamiltonian is

$$H = -\sum_{\langle i,j \rangle} [J_\perp (S_i^x S_j^x + S_i^y S_j^y) + J_z S_i^z S_j^z]. \tag{7}$$

Here $J_z > 0$ and we will assume that $0 \leq J_\perp \leq J_z$. We also assume that only nearest-neighbour sites interact with each other, so the sum is over all pairs of nearest-neighbour sites (each such pair being counted only once).

(a) Use spin-wave theory to calculate the ground state energy E_0 and the magnon dispersion $\omega_{\boldsymbol{k}}$ (in this analysis, neglect terms describing interactions between magnons).

(b) Show that for small $|\boldsymbol{k}|$, the magnon dispersion can be expressed in terms of two parameters Δ and m as

$$\omega_{\boldsymbol{k}} \approx \Delta + \frac{k^2}{2m} \tag{8}$$

and find expressions for Δ and m. Here Δ is the lower bound on the magnon energy and is called the energy gap, while m can be interpreted as a mass (note that we have set $\hbar = 1$; with $\hbar$ reinstated, the term involving m is seen to be of the familiar kinetic energy form $\hbar^2 k^2/(2m)$).

(c) What is the value of Δ in the limit $J_\perp \to J_z$? Is this what you would expect in view of Goldstone's theorem? Explain.

Problem 3

In this problem you will consider electrons in a disordered potential, as discussed in the lectures. The Hamiltonian is given by

$$H = H_0 + V \tag{9}$$

where

$$H_0 = \sum_{\boldsymbol{k}} \xi_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}} c_{\boldsymbol{k}} \tag{10}$$

(we drop the spin index as in the lectures) and the interaction between the electrons and the impurities is described by

$$V = \sum_{\boldsymbol{k},\boldsymbol{k}'} U(\boldsymbol{k}' - \boldsymbol{k}) \rho(\boldsymbol{k}' - \boldsymbol{k}) \hat{c}^\dagger_{\boldsymbol{k}'} \hat{c}_{\boldsymbol{k}}, \tag{11}$$

where

$$U(\boldsymbol{k}) = \frac{1}{\Omega} \int d\boldsymbol{r}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\, U(\boldsymbol{r}), \tag{12}$$

$$\rho(\boldsymbol{k}) = \sum_{j=1}^{N} e^{-i\boldsymbol{k}\cdot\boldsymbol{R}_j}, \tag{13}$$

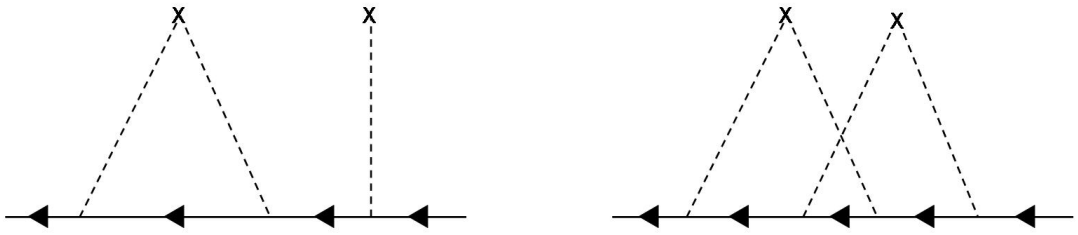
where Ω is the volume of the system, $U(\boldsymbol{r} - \boldsymbol{R}_j)$ is the potential an electron at position $\boldsymbol{r}$ experiences due to the j'th impurity at position $\boldsymbol{R}_j$, and N is the number of impurities. In the lectures we developed a perturbation expansion for the single-particle Matsubara Green function $\mathcal{G}(\boldsymbol{k}, \boldsymbol{k}'; ip_m)$ where p_m is a fermionic Matsubara frequency. Upon averaging over the positions of the impurities, the resulting Green function became $\boldsymbol{k}$-diagonal: $\bar{\mathcal{G}}(\boldsymbol{k}, \boldsymbol{k}'; ip_m) = \bar{\mathcal{G}}(\boldsymbol{k}, ip_m)\delta_{\boldsymbol{k},\boldsymbol{k}'}$. We represented each term in the perturbation expansion for $\bar{\mathcal{G}}(\boldsymbol{k}, ip_m)$ by a Feynman diagram and established the Feynman rules for translating between the diagrams and their associated mathematical expressions.

(a) Suppose that you are presented with an arbitrary Feynman diagram in the perturbation expansion for $\bar{\mathcal{G}}(\boldsymbol{k}, ip_m)$. How would you identify

1. its order n (with respect to the impurity potential).
2. its dependence on the density of impurities $n_{\text{imp}} = N/\Omega$.

(b) Consider the two Feynman diagrams below that appear in the perturbation expansion for $\bar{\mathcal{G}}(\boldsymbol{k}, ip_m)$.

For each diagram:



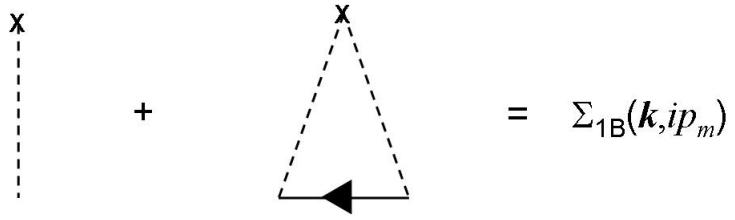
1. Give its mathematical expression (do not attempt to evaluate any wavevector sums).

2. Determine whether the diagram is reducible or irreducible (justify your conclusion). If the diagram is irreducible, draw the corresponding self-energy diagram and give its mathematical expression.

(c) In the lectures we showed that $\bar{\mathcal{G}}(\boldsymbol{k}, ip_m)$ can be expressed as

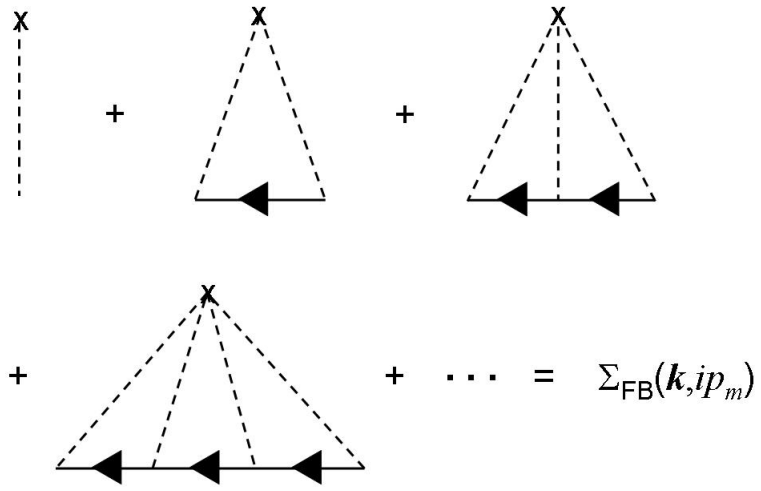
$$\bar{\mathcal{G}}(\boldsymbol{k}, ip_m) = \frac{1}{(\mathcal{G}^{(0)}(\boldsymbol{k}, ip_m))^{-1} - \Sigma(\boldsymbol{k}, ip_m)} = \frac{1}{ip_m - \xi_{\boldsymbol{k}} - \Sigma(\boldsymbol{k}, ip_m)} \qquad (14)$$

where $\Sigma(\boldsymbol{k}, ip_m)$ is the self-energy, defined as the sum of all self-energy diagrams. We found an approximate result for $\bar{\mathcal{G}}(\boldsymbol{k}, ip_m)$ by approximating the self-energy $\Sigma(\boldsymbol{k}, ip_m)$ as



This is called the "first Born approximation," which is why we have here defined the sum of these two self-energy diagrams as $\Sigma_{\rm 1B}(\boldsymbol{k}, ip_m)$. In the following we will study a different approximation to the self-energy, given by the infinite sum of *all* self-energy diagrams with a *single* impurity cross (see the figure below).

This is called the "full Born approximation," which is why we have defined this sum as $\Sigma_{\rm FB}(\boldsymbol{k}, ip_m)$. Note that the first two terms in this infinite sum constitute the "first Born approximation" $\Sigma_{\rm 1B}(\boldsymbol{k}, ip_m)$.



1. Explain under what physical conditions (specified by the impurity density $n_{\rm imp}$ and the strength of the scattering potential $U(\boldsymbol{r})$) you expect $\Sigma_{\rm 1B}$ to be a good approximation to the full self-energy. Do the same for $\Sigma_{\rm FB}$. Justify your answers.

2. Give the mathematical expression for the nth diagram in $\Sigma_{\rm FB}(\boldsymbol{k}, ip_m)$, i.e. the diagram with n interaction lines where n is an arbitrary positive integer. (You may find it helpful to first consider the expressions for the first few diagrams in $\Sigma_{\rm FB}(\boldsymbol{k}, ip_m)$ before you consider the case of a general n.)

3. Assume that the impurity potential $U(\boldsymbol{r})$ is very short-ranged, so that its Fourier transform $U(\boldsymbol{k})$ can be approximated by a constant U, i.e. $U(\boldsymbol{k}) \equiv U$ for all $\boldsymbol{k}$. Show that in this case, $\Sigma_{\rm FB}(\boldsymbol{k}, ip_m)$ is given by

$$\Sigma_{\rm FB}(\boldsymbol{k}, ip_m) = \frac{NU}{1 - U\sum_{\boldsymbol{k}_1} \mathcal{G}^{(0)}(\boldsymbol{k}_1, ip_m)} \equiv \Sigma_{\rm FB}(ip_m). \tag{15}$$

4. In the lectures we found that for a very short-ranged impurity potential, the second diagram in $\Sigma_{\rm 1B}$ and $\Sigma_{\rm FB}$ in the figures above is given by

$$-\frac{i}{2\tau_{\rm 1B}}\mathrm{sgn}(p_m) \quad \text{where} \quad \frac{1}{\tau_{\rm 1B}} = 2\pi n_{\rm imp} u^2 D(0). \tag{16}$$

   (In the lectures $\tau_{\rm 1B}$ was just called τ.) Here $u = U\Omega$ and $D(0)$ is a density-of-states factor whose precise definition is unimportant here. Use the result (16) to show that

$$\mathrm{Im}\,\Sigma_{\rm FB}(ip_m) = -\frac{1}{2\tau_{\rm FB}}\mathrm{sgn}(p_m) \tag{17}$$

   and give an expression for the parameter $\tau_{\rm FB}$.

Formulas

Trigonometric identities:

$$\cos 2x = \cos^2 x - \sin^2 x, \tag{18}$$

$$\sin 2x = 2 \sin x \cos x, \tag{19}$$

$$\cos^2 x = \frac{1}{1 + \tan^2 x} \tag{20}$$

Spin interactions:

$$S_i^x S_j^x + S_i^y S_j^y = \frac{1}{2}(S_i^+ S_j^- + S_i^- S_j^+)$$

Holstein-Primakoff representation:

$$\begin{aligned} S_j^+ &= \sqrt{2S - n_j}\, a_j, \\ S_j^- &= a_j^\dagger \sqrt{2S - n_j}, \\ S_j^z &= S - n_j, \end{aligned}$$

where $n_j \equiv a_j^\dagger a_j$

$$a_j = \frac{1}{\sqrt{N}} \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}_j} a_{\boldsymbol{k}}$$

Lattice sum:

$$\frac{1}{N} \sum_j e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}_j} = \delta_{\boldsymbol{k},\boldsymbol{k}'}$$

Geometric series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$