NTNU
The Faculty of Science and Technology
Department of Physics

Exam TFY 4210 Quantum theory of many-particle systems, spring 2012

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Monday, 4 June, 2012
09.00-13.00h

Examination support:

Approved calculator
Rottmann: Matematisk Formelsamling
Rottmann: Matematische Formelsammlung
Barnett & Cronin: Mathematical Formulae

The exam has 3 problems, with subproblems (a), (b), ... All subproblems have the same weight. There are 6 pages in total. Some useful formulas are given on the last page .

Problem 1

(a) The Dirac equation reads (with $\hbar = c = 1$)

$$i\frac{\partial\psi}{\partial t} = H\psi \quad \text{where } H = \vec{\alpha}\cdot\vec{p} + \beta m.$$

Briefly describe why Dirac sought an equation of this form.

(b) It turns out that an equation of this form also arises in the low-energy description of some 1-dimensional condensed matter systems. In the rest of this problem we therefore consider the Dirac equation in 1 spatial dimension. There is then only one α matrix, α_1. Use the same kind of reasoning as for the 3-dimensional case to show that in the 1-dimensional case one gets the conditions

$$\alpha_1^2 = \beta^2 = 1,$$
$$\alpha_1\beta + \beta\alpha_1 = 0.$$

(c) A valid representation for β and α_1 that satisfies these equations is $\beta = \sigma_1$ and $\alpha_1 = \sigma_3$. Using this Pauli matrix representation, show that the eigenvalues of H are given by

$$E = \pm\sqrt{p^2 + m^2}$$

where p is the momentum eigenvalue.

(d) In terms of γ matrices ($\gamma^0 \equiv \beta$ and $\gamma^1 \equiv \beta\alpha_1$) the Dirac equation reads

$$(i\gamma^\mu\partial_\mu - m)\psi = 0.$$

where μ runs over 0 and 1. Derive this equation from the Lagrangian density

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

where $\bar{\psi} = \psi^\dagger\gamma^0$.

(e) With our chosen representations for β and α_1, the γ matrices become $\gamma^0 = \sigma_1$ and $\gamma^1 = -i\sigma_2$. Consider the matrix $\gamma^5 \equiv \gamma^0\gamma^1$, which is used to define a *chiral transformation* as

$$\psi \to e^{i\theta\gamma^5}\psi$$

where θ is an angular parameter. Show that under this transformation, $\bar{\psi}$ transforms as

$$\bar{\psi} \to \bar{\psi}\, e^{i\theta\gamma^5},$$

and show furthermore that the two-component vector

$$\begin{pmatrix} \bar{\psi}\psi \\ i\bar{\psi}\gamma^5\psi \end{pmatrix}$$

transforms as a rotation,

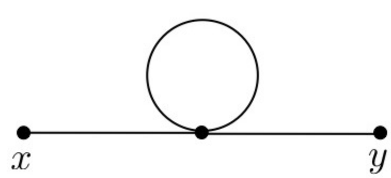
$$\begin{pmatrix} \bar{\psi}\psi \\ i\bar{\psi}\gamma^5\psi \end{pmatrix} \to \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \bar{\psi}\psi \\ i\bar{\psi}\gamma^5\psi \end{pmatrix}$$

where the rotation angle $\phi = \phi(\theta)$. Find the values of θ that leave this two-component vector invariant. (These results have a natural interpretation in the condensed matter context that we mentioned in the introduction, but we don't go into that here.)

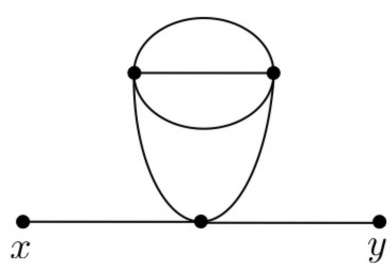
Problem 2

In this problem we consider φ^4 quantum field theory. Subproblems (a) and (b) are about (position-space) Feynman diagrams for the 2-point function $\langle\Omega|T\{\varphi(x)\varphi(y)\}|\Omega\rangle \equiv D_F(x-y)_{\text{int}}$ in φ^4 theory. Subproblem (c) involves (momentum-space) Feynman diagrams for the Fourier transform $\widetilde{D}_F(p)_{\text{int}}$ of the 2-point function.

(a) Using the Feynman rules for position-space Feynman diagrams, write down the expression for the two Feynman diagrams (i)-(ii) below (you can leave the symmetry factor S unspecified).

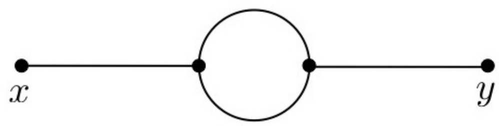


(i)

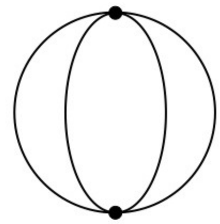


(ii)

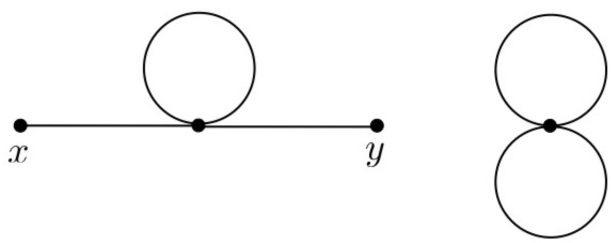
(b) After some simplifications, the perturbation expansion for the 2-point function can be written schematically as a sum over Feynman diagrams, i.e. $D_F(x-y)_{\text{int}} = \sum_i A_i$, where A_i represents a Feynman diagram appearing in this expansion. Among the 4 diagrams (i)-(iv) below, at least one of them is not of the valid type A_i. Identify the invalid diagram(s), and if a diagram is invalid, briefly state why.



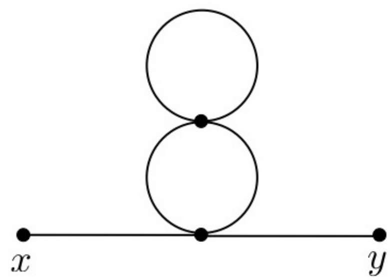
(i)



(ii)

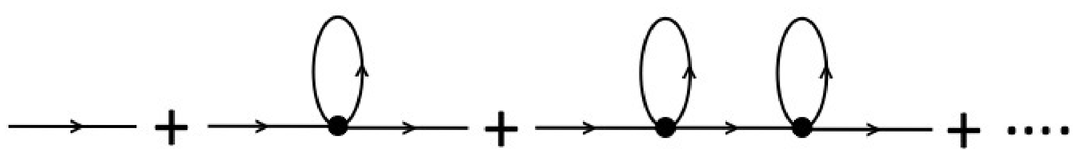


(iii)



(iv)

(c) Consider the following approximation for $\tilde{D}_F(p)_{\text{int}}$:



Using the momentum-space Feynman rules, find an expression for the diagram with n loops in this series. [Hint: It may be helpful to start by finding expressions for the diagrams with 0, 1, and 2 loops, and then if necessary look at diagrams with more loops until you see a pattern. Note that the symmetry factor for the diagram with n loops is 2^n.] Use this to find an expression for $\tilde{D}_F(p)_{\text{int}}$ in this approximation. (Don't try to evaluate nontrivial integrals.)

Problem 3

Consider a tight-binding model of noninteracting electrons in a one-dimensional crystal with N sites and periodic boundary conditions. The Hamiltonian is

$$H = -t \sum_{j,\sigma} (c^\dagger_{j,\sigma} c_{j+1,\sigma} + \text{h.c.}) + t' \sum_{j,\sigma} (c^\dagger_{j,\sigma} c_{j+2,\sigma} + \text{h.c.}).$$

Here $c^\dagger_{j,\sigma}$ ($c_{j,\sigma}$) creates (annihilates) an electron with spin projection σ ($= \pm 1/2$) on site j. The first (second) term in H describes hopping between nearest-neighbour (next-nearest-neighbour) sites. These terms have hopping amplitudes $-t$ and t', respectively.

(a) Show that H can be written on diagonal form as

$$H = \sum_{k,\sigma} \varepsilon_k c^\dagger_{k\sigma} c_{k\sigma}$$

where $c^\dagger_{k,\sigma}$ ($c_{k,\sigma}$) creates (annihilates) an electron with wavevector k and spin projection σ, the k sum is over the 1st Brillouin zone $[-\pi, \pi\rangle$ and

$$\varepsilon_k = -2t \cos k + 2t' \cos 2k$$

(the wavevectors are dimensionless as we have set the lattice spacing to 1).

From now on, assume that t is positive and that the system is *half-filled*, i.e. the number of electrons N_e equals the number of sites N. We will consider the ground state of the Hamiltonian for different nonnegative values of t'. To be precise we define here a Fermi wavevector of a one-dimensional system as a wavevector that separates a region of occupied wavevectors from a region of unoccupied wavevectors in the ground state of the system.

(b) First consider the case $t' = 0$. Sketch ε_k. What are the values of the Fermi wavevectors and the occupied wavevectors?

(c) Next consider t' to be positive and define the ratio $r = t'/t$ (> 0). Show that there is a critical value r_c such that for $r < r_c$ the system has two Fermi wavevectors while for $r > r_c$ the system has four Fermi wavevectors. Derive the value of r_c and find the Fermi energy at $r = r_c$.

Formulas

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{i,j} \qquad (i, j = 1, 2, 3)$$

$$\tilde{D}_F(p) = \frac{i}{p^2 - m^2 + i\epsilon}$$

$$\frac{1}{N} \sum_j e^{i(k-k')j} = \delta_{k,k'}$$