NTNU
Faculty for Science and Technology
Department of Physics

Exam TFY 4210 Quantum Theory for Many-particle Systems spring 2011

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09.00-13.00h

Examination support:
Approved calculator
Rottmann: Matematisk Formelsamling
Rottmann: Matematische Formelsammlung
Barnett & Cronin: Mathematical Formulae

The metric is $(1,-1,-1,-1)$ and we use the units $\hbar = c = 1$. Useful formulas can be found in the Appendix. The problem set is four pages. Read carefully. Viel Glück!

Problem 1

Consider Dirac fermions in $2+1$ dimensions, i.e. in the x-y plane. The Lagrangian density is given by

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi\,, \tag{1}$$

where $\mu = 0, 1, 2$ and m is the mass of the fermions. The γ-matrices satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,. \tag{2}$$

In $3+1$ dimensions these matrices are 4×4 matrices, but in in $2+1$ dimensions we can choose 2×2 matrices. In the following, we choose $\gamma^0 = \sigma_3$, $\gamma^1 = i\sigma_2$, and $\gamma^2 = -i\sigma_1$, where σ_i are the Pauli matrices. The metric is $(1,-1,-1)$.

1) Show that the Lagrangian density is invariant under global phase transformations. Is the Lagrangian density invariant under other transformations?

We couple the fermion field to an external gauge field A_μ by replacing the partial derivative by the covariant derivative, i. e. $\partial_\mu \to D_\mu = \partial_\mu - iqA_\mu$, where q is the charge of the fermions.

2) The gauge field is given by $A^\mu = (0, 0, Bx)$, i.e. the gauge field is an external field. Find the fields $\mathbf{E}$ and $\mathbf{B}$. Explain briefly why the x-component of the momentum is not a good quantum number, while the y-component of the momentum is.

3) Show that the Dirac equation can be written as

$$\begin{pmatrix} i\frac{\partial}{\partial t} - m & i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - iqBx \\ -i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - iqBx & -i\frac{\partial}{\partial t} - m \end{pmatrix} \psi = 0 \,. \tag{3}$$

The eigenfunctions $\psi(\mathbf{r}, t)$ can be written as

$$\psi(\vec{r}, t) = e^{-i(Et - p_y y)} \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} , \tag{4}$$

where E is the energy of the eigenstate and p_y is the y-component of the momentum. $f(x)$ and $g(x)$ are functions of x. Use this to find the spectrum, i.e. the energy eigenvalues of E.

Problem 2

Consider a selfinteracting complex scalar field Φ with a chemical potential μ. The chemical potential is introduced by replacing the partial derivative ∂_0 by the covariant derivative $\partial_0 - i\mu$ in the Lagrangian density. This yields

$$\mathcal{L} = (\partial_0 + i\mu)\Phi^*(\partial^0 - i\mu\Phi) + (\partial_i\Phi)^*(\partial^i\Phi) - m^2\Phi^*\Phi - \frac{\lambda}{6}(\Phi^*\Phi)^2 \,. \tag{5}$$

1) Is the Lagrangian density Lorentz invariant?

The field Φ is now written as a sum of a classical field ϕ_0 and a fluctuating quantum field as

$$\Phi = \frac{1}{\sqrt{2}}\left(\phi_0 + \phi_1 + i\phi_2\right) . \tag{6}$$

2) Show that the classical potential is

$$V_0 = \frac{1}{2}(m^2 - \mu^2)\phi_0^2 + \frac{\lambda}{24}\phi_0^4 . \tag{7}$$

3) In the remainder $m^2 > 0$. Find the global minimum of the potential V_0 as a function of μ. Draw a figure and explain.

4) The quadratic term in $\mathcal{L}$ can be written as

$$\mathcal{L}_{\rm kvad} = -\frac{1}{2}(\phi_1, \phi_2)\left(\begin{array}{cc} \partial_0^2 - \nabla^2 + M_1^2 & -2\mu\partial_0 \\ 2\mu\partial_0 & \partial_0^2 - \nabla^2 + M_2^2 \end{array}\right)\left(\begin{array}{c} \phi_1 \\ \phi_2 \end{array}\right) , \tag{8}$$

where $M_1^2 = m^2 - \mu^2 + \frac{\lambda}{2}\phi_0^2$ and $M_2^2 = m^2 - \mu^2 + \frac{\lambda}{6}\phi_0^2$. Use this to find the dispersion relations as a function of μ. Explain the results.

Problem 3

Consider bosons and Bose-Einstein condensation in two spatial dimensions

1) The free energy density $\mathcal{F}$ is given by

$$\mathcal{F} = -\frac{\mu^2}{2g} + \frac{1}{2}\int \frac{d^2p}{(2\pi)^2}\epsilon(p) , \tag{9}$$

where μ is the chemical potential, g is the coupling constant, and $\epsilon(p)$ is the Bogoliubov dispersion relation

$$\epsilon(p) = \sqrt{\frac{p^2}{2m}\left(\frac{p^2}{2m} + 2\mu\right)} , \tag{10}$$

wheere m is the mass of the bosons.

1) Explain briefly the various terms in the expression for $\mathcal{F}$. Show that the renormalized free energy $\mathcal{F}$ can be written as

$$\mathcal{F} = -\frac{\mu^2}{2g} - \frac{m\mu^2}{8\pi}\left[\ln\frac{\Lambda^2}{\mu} + C\right] . \tag{11}$$

2) Use the above result to calculate the density as a function of μ and the energy density $\mathcal{E}$ as a function of ρ.

3) We have used dimensional regularization. Are there other ways of regularizing divergent integrals? What are the advantages of dimensional regularization.

Useful formulas:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

$$\begin{aligned}
\Lambda^{2\epsilon} \int \frac{d^d p}{(2\pi)^d} p \sqrt{p^2 + y^2} &= -\frac{y^4}{32\pi} \left[\frac{1}{\epsilon} + \log \frac{\Lambda^2}{m\mu} + C + \mathcal{O}(\epsilon) \right] , \\
\rho &= -\frac{\partial \mathcal{F}}{\partial \mu} \\
\mathcal{E} &= \mathcal{F} + \rho\mu , \\
\mathbf{E} &= -\nabla A^0 + \frac{\partial \mathbf{A}}{\partial t} , \\
\mathbf{B} &= \nabla \times \mathbf{A} .
\end{aligned}$$