

Introducing the Fourier transform $\mathcal{G}_{k_b k_a}^{(n)}$ of $\mathcal{G}^{(n)}(\mathbf{r}_b, \mathbf{r}_a)$ as

$$\mathcal{G}^{(n)}(\mathbf{r}_b, \mathbf{r}_a) = \frac{1}{V^2} \sum_{k_a k_b} e^{ik_b \cdot \mathbf{r}_b} e^{-ik_a \cdot \mathbf{r}_a} \mathcal{G}_{k_b k_a}^{(n)}, \quad (12.28)$$

with

$$\begin{aligned} \mathcal{G}_{k_b k_a}^{(n)} &= \sum_{j_1 \dots j_n} \frac{1}{V^{n-1}} \sum_{k_1 \dots k_{n-1}} e^{-i[(k_b - k_{n-1}) \cdot \mathbf{P}_{j_n} + \dots + (k_1 - k_n) \cdot \mathbf{P}_{j_1}]} \\ &\quad \times \mathcal{G}_{k_b}^0 u_{k_b - k_{n-1}} \mathcal{G}_{k_{n-1}}^0 \dots u_{k_2 - k_1} \mathcal{G}_{k_1}^0 \dots u_{k_1 - k_n} \mathcal{G}_{k_n}^0. \end{aligned} \quad (12.29)$$

We can now easily deduce the Feynman rules for the diagrams corresponding to $\mathcal{G}_{k_b k_a}^{(n)}$:

- (1) Let dashed arrows $j \bullet \cdots \bullet \star q, \mathbf{P}_j$ denote a scattering event $u_q e^{-iq \cdot \mathbf{P}_j}$.
- (2) Draw n scattering events.
- (3) Let straight arrows $\longrightarrow k$ denote $\mathcal{G}_k^0$.
- (4) Let $\mathcal{G}_{k_a}^0$ go into vertex $\bullet 1$ and $\mathcal{G}_{k_b}^0$ away from vertex $\bullet n$.
- (5) Let $\mathcal{G}_{k_j}^0$ go from vertex j to vertex $j+1$.
- (6) Maintain momentum conservation at each vertex.
- (7) Perform the sums $\frac{1}{V} \sum_{k_j}$ over all internal momenta k_j , and $\sum_{j_1 \dots j_n}^{N_{\text{imp}}}$ over $\mathbf{P}_{j_i}$.

(12.30)

The diagram corresponding to Eq. (12.29) is

$$\mathcal{G}_{k_b k_a}^{(n)} = \begin{array}{c} \star \mathbf{P}_n \\ | \\ \bullet 1 \quad k_b - k_{n-1} \\ | \\ \bullet 2 \quad k_3 - k_2 \\ | \\ \bullet 3 \quad k_2 \\ | \\ \bullet 4 \quad k_{n-1} \\ | \\ \bullet n \quad k_b \end{array} \longrightarrow \cdots \longrightarrow \begin{array}{c} \star \mathbf{P}_1 \\ | \\ \bullet 1 \quad k_1 - k_n \\ | \\ \bullet 2 \quad k_2 \\ | \\ \bullet 3 \quad k_3 \\ | \\ \bullet n \quad k_a \end{array} \quad (12.31)$$

This diagram is very suggestive. One can see how an incoming electron with momentum k_a is scattered n times under momentum conservation with the impurities and leaves the system with momentum k_b . However, as mentioned above, it is not possible to continue the study of impurity scattering on general grounds without further assumptions. We therefore begin to consider the possibility of performing an average over the random positions $\mathbf{P}_j$ of the impurities.

12.4 Impurity self-average

If the electron wavefunctions are completely coherent throughout the entire disordered metal each true electronic eigenfunction exhibit an extremely complex diffraction pattern spawned by the randomly positioned scatterers. If one imagines changing some external parameter, e.g., the average electron density or an external magnetic field,

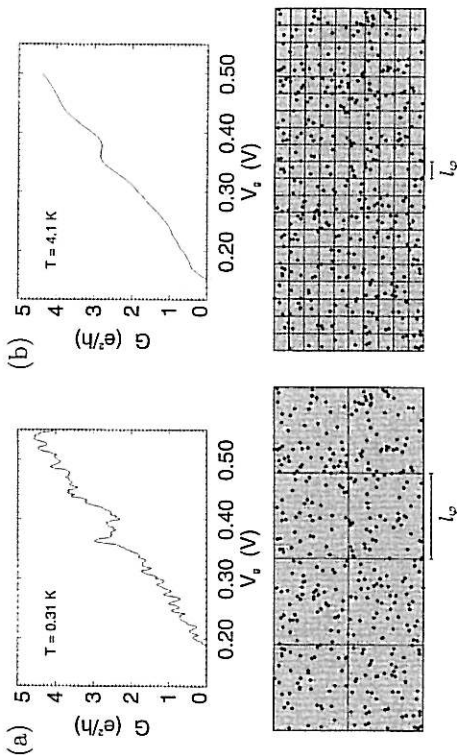


FIG. 12.2. (a) The measured conductance of a disordered GaAs sample at $T = 0.31$ K displaying random but reproducible quantum fluctuations as a function of a gate voltage V_g controlling the electron density. The fluctuations are due to phase coherent scattering against randomly positioned impurities. Below is indicated that the phase coherence length l_ϕ is large compared to the size of the sample. (b) The same system at $T = 4.1$ K. The fluctuations are almost gone due to the smallness of l_ϕ at this temperature. The sample now contains a large number of independent but phase-coherent subsystems of size l_ϕ . As a result a substantial self-averaging occurs, which suppresses the quantum fluctuations.

each individual diffraction pattern will of course change drastically due to the sensitivity of the scattering phases of the wavefunctions. Significant quantum fluctuations must therefore occur in any observable at sufficiently low temperatures.

Using modern nanotechnology to fabricate small (but still macroscopic) samples, and standard cryogenic equipment to cool down these samples to ultralow temperatures, one can in fact obtain an experimental situation where the electrons can traverse the sample without losing their quantum-mechanical phase coherence. In Fig. 12.2(a) is shown the conductance trace of a GaAs nano-device, such as the one shown in Fig. 2.10, at 0.31 K as a function of the electron density. This density can be controlled by applying a gate voltage V_g on an external electrode. The conductance G is seen to fluctuate strongly for minute changes of V_g . These fluctuations turn out to be perfectly reproducible as V_g is swept up and down several times.

As the temperature of a given sample is raised, the amount of electron-electron and electron-phonon scattering increases because of an increased phase space for scattering and an increased number of phonons. The quantum mechanical phase of each individual electron is changed by a small random amount at each inelastic scattering event, and as a result the coherence length l_ϕ for the electrons diminishes. At suffi-

ciently high temperature (e.g., 4.1 K) l_ϕ is much smaller than the size of the device, and we can think of the device as being composed as a number of phase-independent small phase-coherent subsystems. Therefore, when one measures an observable the result is in fact an incoherent average of all these subsystems. Note that this average is imposed by the physical properties of the system itself, and this effective averaging is consequently denoted *self-averaging*. This effect is illustrated in Fig. 12.2(b) where the conductance trace at 4.1 K is seen to be much smoother than the one at 0.31 K, and where the many small phase-coherent subsystems of the sample are indicated below the experimental graph.

For very large (mm-sized) macroscopic samples l_ϕ is much smaller than the sample size at all experimental realizable temperatures ($T > 10$ mK for electron gases in metals and semiconductors), and we are in the impurity self-averaging case. Mathematically, the impurity average is performed by summing over all the phase-independent coherent subsystems and dividing by their number N_{sys} . But due to the random distribution of the impurities, this average is the same as an average over the impurity position within a single subsystem, as can be seen from Fig. 12.2. However, even on the rather small length scale l_ϕ the system is already homogeneous, and one can as well perform the position average over the entire volume of the sample. Thus in the following we average over all possible uncorrelated positions $\mathbf{P}_j$ of the N_{imp} impurities for the entire system:

$$\begin{aligned} \frac{1}{V} \langle G_{\mathbf{k}_b, \mathbf{k}_a} \rangle_{\text{imp}} &\equiv \delta_{\mathbf{k}_b, \mathbf{k}_a} \bar{G}_{\mathbf{k}_a} \equiv \frac{\delta_{\mathbf{k}_b, \mathbf{k}_a}}{N_{\text{sys}}} \sum_{i=1}^{N_{\text{sys}}} G_{\text{sys}_i}^{\text{sys}_i} \\ &\sim \delta_{\mathbf{k}_b, \mathbf{k}_a} \frac{1}{V} \int d\mathbf{P}_1 \frac{1}{V} \int d\mathbf{P}_2 \cdots \frac{1}{V} \int d\mathbf{P}_{N_{\text{imp}}} G_{\mathbf{k}_a} \end{aligned} \quad (12.32)$$

Here we have anticipated that the impurity-averaged Green's function is diagonal in $\mathbf{k}$ due to the restoring of translation invariance upon average. Some care must be taken regarding the average over the impurity positions $\mathbf{P}_j$. Any n th-order contribution to $G_{\mathbf{k}}$ contains n scattering events, but they need not be on n different scatterers. In fact, for any number p , $1 \leq p \leq n$ of scatterers could be involved. We must therefore carefully sort out all possible ways to scatter on p different impurities.

As mentioned in Eq. (12.20) we work in the limit of small impurity densities n_{imp} . For a given fixed number n of scattering events the most important contribution therefore comes from processes involving just one impurity. Then, down by the small factor $n_{\text{imp}}/n_{\text{el}}$, follow processes involving two impurities, etc. We note that in Eq. (12.29) the only reference to the impurity positions is the exponential $\exp[i(\mathbf{q}_1 \cdot \mathbf{P}_1 + \mathbf{q}_2 \cdot \mathbf{P}_2 + \cdots + \mathbf{q}_n \cdot \mathbf{P}_n)]$, with the scattering vectors $\mathbf{q}_i = \mathbf{k}_i - \mathbf{k}_{i-1}$. The sum in Eq. (12.29) over impurity positions in this exponential is now ordered according to how many impurities are involved: