TFY4210, Quantum theory of many-particle systems, 2013: Tutorial 7

Consider the Hamiltonian

$$H = \varepsilon(a_1^\dagger a_1 + a_2^\dagger a_2) + \Delta(a_1 a_2 + \text{h.c.}) \tag{1}$$

where a_1 and a_2 are bosonic operators satisfying canonical commutation relations $[a_i, a_j^\dagger] = \delta_{ij}$ etc. $(i, j = 1, 2)$. Assume that ε and Δ are positive numbers with $\varepsilon > \Delta$.

In order to write the Hamiltonian in diagonal form we transform to a new set b_1, b_2 of bosonic operators. The transformation reads (here u and v are real numbers)

$$a_1 = ub_1 - vb_2^\dagger, \tag{2}$$
$$a_2 = ub_2 - vb_1^\dagger. \tag{3}$$

(a) Use the requirement that the b-operators should also satisfy canonical commutation relations to show (e.g. by just calculating one selected commutator) that

$$u^2 - v^2 = 1. \tag{4}$$

This result can be used to write $u = \cosh\eta$, $v = \sinh\eta$.

(b) Show that H becomes diagonal in terms of the b-operators provided that η is chosen to satisfy

$$\tanh 2\eta = \frac{\Delta}{\varepsilon}. \tag{5}$$

Show that with this choice, H can be written

$$H = F(b_1^\dagger b_1 + b_2^\dagger b_2) + G \tag{6}$$

and give expressions for F and G in terms of ε and Δ.

(c) Argue from this result that the ground state $|\Psi_0\rangle$ can be defined in terms of an equation expressing what happens when an annihilation operator b_i acts on $|\Psi_0\rangle$. What is the ground state energy? Explain your reasoning.

(d) What is the energy of the lowest excited state(s)? Explain your reasoning.

(e) Express the annihilation operator b_1 in terms of a-operators.

(f) The ground state $|\Psi_0\rangle$ is given by

$$|\Psi_0\rangle = C\ \exp(-\tanh\eta\ a_1^\dagger a_2^\dagger)|0\rangle \tag{7}$$

where $|0\rangle$ is the vacuum state of the a-operators, i.e. $a_1|0\rangle = a_2|0\rangle = 0$. The constant C is just a normalization factor and can be neglected in the following.

Verify Eq. (7) for the ground state by showing that it satisfies the defining equation for $|\Psi_0\rangle$ discussed in (c) (it's sufficient that you only consider the equation that involves b_1). [Hint: The Baker-Hausdorff theorem

$$e^{-Q}Pe^{Q} = P + [P,Q] + \frac{1}{2!}[[P,Q],Q] + \frac{1}{3!}[[[P,Q],Q],Q] + \dots \tag{8}$$

may be useful.]