TFY4210, Quantum theory of many-particle systems, 2016: Tutorial 7

1. Bogoliubov transformation for bosons.

Consider the Hamiltonian

$$H = \varepsilon(a_1^\dagger a_1 + a_2^\dagger a_2) + \Delta(a_1 a_2 + \text{h.c.}) \tag{1}$$

where a_1 and a_2 are bosonic operators satisfying canonical commutation relations $[a_i, a_j^\dagger] = \delta_{ij}$ etc. $(i, j = 1, 2)$. Assume that ε and Δ are positive numbers with $\varepsilon > \Delta$.

In order to write the Hamiltonian in diagonal form we do a Bogoliubov transformation to a new set b_1, b_2 of bosonic operators. The transformation reads (here u and v are real numbers)

$$a_1 = ub_1 - vb_2^\dagger, \tag{2}$$

$$a_2 = ub_2 - vb_1^\dagger. \tag{3}$$

(a) Use the requirement that the b-operators should also satisfy canonical commutation relations to show (e.g. by just calculating one selected commutator) that

$$u^2 - v^2 = 1. \tag{4}$$

This result can be used to write $u = \cosh\eta$, $v = \sinh\eta$.

(b) Show that H becomes diagonal in terms of the b-operators provided that η is chosen to satisfy

$$\tanh 2\eta = \frac{\Delta}{\varepsilon}. \tag{5}$$

Show that with this choice, H can be written

$$H = F(b_1^\dagger b_1 + b_2^\dagger b_2) + G \tag{6}$$

and give expressions for F and G in terms of ε and Δ.

(c) Argue from this result that the ground state $|\Psi_0\rangle$ can be defined in terms of an equation expressing what happens when an annihilation operator b_i acts on $|\Psi_0\rangle$. What is the ground state energy? Explain your reasoning.

(d) What is the energy of the lowest excited state(s)? Explain your reasoning.

(e) Express the annihilation operator b_1 in terms of a-operators.

(f) The ground state $|\Psi_0\rangle$ is given by

$$|\Psi_0\rangle = C\ \exp(-\tanh\eta\ a_1^\dagger a_2^\dagger)|0\rangle \tag{7}$$

where $|0\rangle$ is the vacuum state of the a-operators, i.e. $a_1|0\rangle = a_2|0\rangle = 0$. The constant C is just a normalization factor and can be neglected in the following.

Verify Eq. (7) for the ground state by showing that it satisfies the defining equation for $|\Psi_0\rangle$ discussed in (c) (it's sufficient that you only consider the equation that involves b_1). [Hint: The Baker-Hausdorff theorem

$$e^{-Q}Pe^{Q} = P + [P,Q] + \frac{1}{2!}[[P,Q],Q] + \frac{1}{3!}[[[P,Q],Q],Q] + \ldots \tag{8}$$

may be useful.]

2. Ferromagnetic Heisenberg model with a spin anisotropy revisited.

Consider again the ferromagnetic Heisenberg model with a spin anisotropy studied in Problem 2 in Tutorial 6, where for $D \geq 0$ it was found that the energy gap $\Delta = 2SD$. Thus the system is gapped for $D > 0$ and gapless for $D = 0$. Discuss this property in light of Goldstone's theorem and the symmetries of the model and its ground states.

3. The total S^z operator for the antiferromagnetic Heisenberg model.

Consider the operator $S^z = \sum_j S_j^z$ representing the total spin in the z direction (the sum is over all N sites in the system).

(a) Use spin-wave theory to show that for the antiferromagnetic Heisenberg model, one can express S^z as

$$S^z = \sum_{\boldsymbol{k}} (\beta_{\boldsymbol{k}}^\dagger \beta_{\boldsymbol{k}} - \alpha_{\boldsymbol{k}}^\dagger \alpha_{\boldsymbol{k}}). \tag{9}$$

(b) Argue that a general eigenstate of the Heisenberg antiferromagnet in the spin-wave approximation can be written (we omit normalization factors)

$$\prod_{\boldsymbol{k}} [(\alpha_{\boldsymbol{k}}^\dagger)^{n_{\alpha_{\boldsymbol{k}}}} (\beta_{\boldsymbol{k}}^\dagger)^{n_{\beta_{\boldsymbol{k}}}}] |G\rangle \tag{10}$$

where $\{n_{\alpha_{\boldsymbol{k}}}\}$ and $\{n_{\beta_k}\}$ are the magnon numbers characterizing the eigenstate and $|G\rangle$ is the ground state discussed in the lectures. Show that this state is also an eigenstate of S^z and determine the corresponding eigenvalue.