TFY4210, Quantum theory of many-particle systems, 2013: Tutorial 8

1. S^z eigenvalues of single-magnon states.

Consider the operator $S^z = \sum_j S_j^z$ representing the total spin in the z direction (the sum is over all the N sites in the system).

(a) For the Heisenberg **ferromagnet**, show that

(i)

$$S^z = NS - \sum_{\boldsymbol{k}} a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}}, \tag{1}$$

(ii) the state $a_{\boldsymbol{k}}^\dagger|0\rangle$ is an eigenstate of S^z with eigenvalue $NS - 1$. (Here $|0\rangle$ is the ferromagnetic ground state considered in the lectures, i.e. with order along the z direction.)

It follows that creation of a ferromagnetic magnon is accompanied by a change $\Delta S^z = -1$ in the z component of the total spin.

(b) For the Heisenberg **antiferromagnet**, show that

(i)

$$S^z = \sum_{\boldsymbol{k}} (b_{\boldsymbol{k}}^\dagger b_{\boldsymbol{k}} - a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}}) = \sum_{\boldsymbol{k}} (\beta_{\boldsymbol{k}}^\dagger \beta_{\boldsymbol{k}} - \alpha_{\boldsymbol{k}}^\dagger \alpha_{\boldsymbol{k}}), \tag{2}$$

(ii) the states $\alpha_{\boldsymbol{k}}^\dagger|0\rangle$ and $\beta_{\boldsymbol{k}}^\dagger|0\rangle$ are eigenstates of S^z with eigenvalues -1 and $+1$, respectively. (Here $|0\rangle$ is the antiferromagnetic ground state considered in the lectures.).

It follows that creation of an antiferromagnetic magnon is accompanied by a change $\Delta S^z = \pm 1$ in the z component of the total spin.

2. Translational invariance.

Consider a system of N electrons in a box of volume Ω. We will write the field operator $\hat{\psi}(x)$ as $\hat{\psi}_s(\boldsymbol{r})$ where $\boldsymbol{r}$ is the position and $s = \pm 1/2$ is the spin projection along the z axis. The field operator can be expanded as

$$\hat{\psi}_s(\boldsymbol{r}) = \frac{1}{\sqrt{\Omega}} \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \hat{c}_{\boldsymbol{k}s} \tag{3}$$

where the operator $\hat{c}_{\boldsymbol{k}s}$ annihilates an electron in a plane-wave state with wavevector $\boldsymbol{k}$ and spin projection s.

(a) Show that

$$\hat{U}^\dagger(\boldsymbol{a})\,\hat{\psi}_\sigma(\boldsymbol{r})\,\hat{U}(\boldsymbol{a}) = \hat{\psi}_\sigma(\boldsymbol{r}-\boldsymbol{a}) \tag{4}$$

where the unitary operator

$$\hat{U}(\boldsymbol{a}) = e^{-i\hat{\boldsymbol{P}}\cdot\boldsymbol{a}/\hbar} \tag{5}$$

generates translations in space by a vector $\boldsymbol{a}$, with

$$\hat{\boldsymbol{P}} = \sum_{\boldsymbol{k},\sigma} \hbar\boldsymbol{k}\;\hat{c}^\dagger_{\boldsymbol{k}\sigma}\hat{c}_{\boldsymbol{k}\sigma}. \tag{6}$$

being the momentum operator.

(b) Consider a system of interacting electrons with Hamiltonian

$$\begin{aligned}\hat{H} &= \sum_\sigma \int d^3r\,\hat{\psi}^\dagger_\sigma(\boldsymbol{r})\left(-\frac{\hbar^2}{2m}\right)\nabla^2\hat{\psi}_\sigma(\boldsymbol{r}) \\ &+ \sum_{\sigma,\sigma'}\int d^3r\int d^3r'\;\hat{\psi}^\dagger_\sigma(\boldsymbol{r})\hat{\psi}^\dagger_{\sigma'}(\boldsymbol{r}')\frac{e^2}{4\pi\epsilon_0|\boldsymbol{r}-\boldsymbol{r}'|}\hat{\psi}_{\sigma'}(\boldsymbol{r}')\hat{\psi}_\sigma(\boldsymbol{r}).\end{aligned} \tag{7}$$

An operator $\hat{O}$ is said to be invariant under spatial translations (translationally invariant for short) if $\hat{U}^\dagger(\boldsymbol{a})\,\hat{O}\,\hat{U}(\boldsymbol{a}) = \hat{O}$ for an arbitrary vector $\boldsymbol{a}$. Use the result in (a) to show that $\hat{H}$ is translationally invariant. Also comment on the case of a modified Hamiltonian containing an additional term of external-potential form given by

$$\sum_\sigma \int d^3r\,\hat{\psi}^\dagger_\sigma(\boldsymbol{r})u(\boldsymbol{r})\hat{\psi}_\sigma(\boldsymbol{r}). \tag{8}$$