TFY4210, Quantum theory of many-particle systems, 2015: Tutorial 8

1. The sublattice magnetization correction for the Heisenberg antiferromagnet at nonzero temperature.

(a) The dispersion relation for antiferromagnetic magnons is $\omega_{\boldsymbol{k}} = JSz\sqrt{1-\gamma_{\boldsymbol{k}}^2}$ where $\gamma_{\boldsymbol{k}} = (2/z)\sum_{\boldsymbol{\delta}} \cos(\boldsymbol{k}\cdot\boldsymbol{\delta})$ and $z = 2d$ is the number of nearest neighbour sites. Show that for small $|\boldsymbol{k}|$,

$$\gamma_{\boldsymbol{k}} \approx 1 - \frac{\boldsymbol{k}^2}{2d}, \tag{1}$$

$$\omega_{\boldsymbol{k}} \approx 2JS\sqrt{d}|\boldsymbol{k}|. \tag{2}$$

In the lectures we found that the temperature-dependent part of the sublattice magnetization correction ΔM_A is given by

$$\frac{2}{N}\sum_{\boldsymbol{k}} n_{\boldsymbol{k}} \frac{1}{\sqrt{1-\gamma_{\boldsymbol{k}}^2}} \tag{3}$$

where $n_{\boldsymbol{k}} = \frac{1}{e^{\omega_{\boldsymbol{k}}/k_BT}-1}$.

(b) Show that in $d=1$ and $d=2$ the contribution from small $\boldsymbol{k}$ makes (3) diverge. Hence there is no magnetic order in the Heisenberg antiferromagnet at finite temperatures in one and two dimensions.

(c) In contrast, show that in $d=3$ the expression (3) is finite (i.e. there is no divergence at small $\boldsymbol{k}$) and scales as T^2.

2. 0th and 1st order perturbation theory for the interacting electron gas.

Consider the 3-dimensional interacting electron gas (more precisely the so-called jellium model introduced in Problem 2 in Tutorial 3) with Hamiltonian

$$H = \sum_{\boldsymbol{k},\sigma} \frac{\hbar^2 k^2}{2m} c_{\boldsymbol{k}\sigma}^\dagger c_{\boldsymbol{k}\sigma} + \frac{1}{2\Omega}\sum_{\boldsymbol{q}\neq 0}\sum_{\boldsymbol{k},\sigma}\sum_{\boldsymbol{k}',\sigma'} \frac{e^2}{\varepsilon_0 q^2} c_{\boldsymbol{k}+\boldsymbol{q},\sigma}^\dagger c_{\boldsymbol{k}'-\boldsymbol{q},\sigma'}^\dagger c_{\boldsymbol{k}',\sigma'} c_{\boldsymbol{k},\sigma}. \tag{4}$$

It will be convenient to introduce the length scale defined by the Bohr radius $a_B = \frac{4\pi\varepsilon_0\hbar^2}{me^2}$ and the energy scale $\mathrm{Ry} = \frac{\hbar^2}{2ma_B^2}$ (the Rydberg). Let r_0 be a measure of the average distance between electrons (defined as the radius of a sphere whose volume equals the volume per electron) and define the dimensionless quantity $r_s \equiv r_0/a_B$.

(a) First consider the noninteracting electron gas, whose Hamiltonian is given by the kinetic energy term only. Its ground state is the filled Fermi sphere $|\mathrm{FS}\rangle$ with radius k_F. Show that

$$k_F a_B = \left(\frac{9\pi}{4}\right)^{1/3} \frac{1}{r_s} \tag{5}$$

and that the ground state energy per particle is given by

$$\frac{E^{(0)}}{N} = \frac{3}{5}(k_F a_B)^2 \text{ Ry} \approx \frac{2.21}{r_s^2} \text{ Ry.} \tag{6}$$

(Here you may make use of results already derived in the lectures for $E^{(0)}/N$ and the relation between k_F and the electron density.)

(b) Next consider the interaction term in (4) as a perturbation on the kinetic energy term. Show that the 1st order correction to the ground state energy per particle is given by

$$\frac{E^{(1)}}{N} = -\frac{3}{2\pi}(k_F a_B) \text{ Ry} \approx -\frac{0.916}{r_s} \text{ Ry.} \tag{7}$$

[A few hints: Note that $\boldsymbol{q} \neq 0$ in the interaction term and show that therefore

$$\langle \mathrm{FS}| c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\sigma} c^\dagger_{\boldsymbol{k}'-\boldsymbol{q},\sigma'} c_{\boldsymbol{k}',\sigma'} c_{\boldsymbol{k},\sigma} |\mathrm{FS}\rangle = -\delta_{\boldsymbol{k}',\boldsymbol{k}+\boldsymbol{q}} \delta_{\sigma,\sigma'} \theta(k_F - |\boldsymbol{k}+\boldsymbol{q}|)\theta(k_F - |\boldsymbol{k}|)$$

where $\theta(x)$ is the (Heaviside) step function. Convert the sums over $\boldsymbol{k}$ and $\boldsymbol{q}$ to integrals over spherical coordinates. Observe that for a fixed $\boldsymbol{q}$ the $\boldsymbol{k}$-integration amounts to finding the volume of the intersection of two spheres of radius k_F displaced from each other by a vector $\boldsymbol{q}$.]

3. Single-particle retarded Green function for noninteracting bosons.

Calculate the Fourier transform of the single-particle retarded Green function, $G_0^R(\nu, \omega)$, for noninteracting bosons with Hamiltonian

$$H_0 = \sum_\nu \xi_\nu c^\dagger_\nu c_\nu. \tag{8}$$

4. The basis invariance of the trace.

Show that the trace of an operator is independent of the basis chosen to evaluate it. [Hint: First define Tr O as the sum of the diagonal elements of O in some particular, but arbitrarily chosen basis. Then do a transformation to an arbitrary different basis and show that Tr O can be rewritten as the sum of the diagonal elements of O in the new basis.]