TFY4210, Quantum theory of many-particle systems, 2016: Tutorial 8

1. The sublattice magnetization correction for the Heisenberg antiferromagnet at nonzero temperature.

(a) The dispersion relation for antiferromagnetic magnons is $\omega_{\mathbf{k}} = JSz\sqrt{1-\gamma_{\mathbf{k}}^2}$ where $\gamma_{\mathbf{k}} = (2/z)\sum_{\boldsymbol{\delta}}\cos(\mathbf{k}\cdot\boldsymbol{\delta})$ and z = 2d is the number of nearest neighbour sites. Show that for small $|\mathbf{k}|$,

$$\gamma_{k} \approx 1 - \frac{k^2}{2d},$$
 (1)

$$\omega_{\boldsymbol{k}} \approx 2JS\sqrt{d}|\boldsymbol{k}|. \tag{2}$$

In the lectures we found that the temperature-dependent part of the sublattice magnetization correction ΔM_A is given by

$$\frac{2}{N}\sum_{\boldsymbol{k}} n_{\boldsymbol{k}} \frac{1}{\sqrt{1-\gamma_{\boldsymbol{k}}^2}} \tag{3}$$

where $n_{\mathbf{k}} = \frac{1}{e^{\omega_{\mathbf{k}}/k_B T} - 1}$.

(b) Show that in d = 1 and d = 2 the contribution from small k makes (3) diverge. Hence there is no magnetic order in the Heisenberg antiferromagnet at finite temperatures in one and two dimensions.

(c) In contrast, show that in d = 3 the expression (3) is finite (i.e. there is no divergence at small \mathbf{k}) and scales as T^2 .

2. 0th and 1st order perturbation theory for the interacting electron gas.

Consider the 3-dimensional interacting electron gas (more precisely the so-called jellium model introduced in Problem 2 in Tutorial 3) with Hamiltonian

$$H = \sum_{\boldsymbol{k},\sigma} \frac{\hbar^2 k^2}{2m} c^{\dagger}_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} + \frac{1}{2\Omega} \sum_{\boldsymbol{q}\neq 0} \sum_{\boldsymbol{k},\sigma} \sum_{\boldsymbol{k}',\sigma'} \frac{e^2}{\varepsilon_0 q^2} c^{\dagger}_{\boldsymbol{k}+\boldsymbol{q},\sigma} c^{\dagger}_{\boldsymbol{k}'-\boldsymbol{q},\sigma'} c_{\boldsymbol{k}',\sigma'} c_{\boldsymbol{k},\sigma}.$$
 (4)

It will be convenient to introduce the length scale defined by the Bohr radius $a_B = \frac{4\pi\varepsilon_0\hbar^2}{me^2}$ and the energy scale $\text{Ry} = \frac{\hbar^2}{2ma_B^2}$ (the Rydberg). Let r_0 be a measure of the average distance between electrons (defined as the radius of a sphere whose volume equals the volume per electron) and define the dimensionless quantity $r_s \equiv r_0/a_B$. (a) First consider the noninteracting electron gas, whose Hamiltonian is given by the kinetic energy term only. Its ground state is the filled Fermi sphere $|FS\rangle$ with radius k_F . Show that

$$k_F a_B = \left(\frac{9\pi}{4}\right)^{1/3} \frac{1}{r_s} \tag{5}$$

and that the ground state energy per particle is given by

$$\frac{E^{(0)}}{N} = \frac{3}{5} (k_F a_B)^2 \text{ Ry} \approx \frac{2.21}{r_s^2} \text{ Ry.}$$
(6)

(Here you may make use of results already derived in the lectures for $E^{(0)}/N$ and the relation between k_F and the electron density.)

(b) Next consider the interaction term in (4) as a perturbation on the kinetic energy term. Show that the 1st order correction to the ground state energy per particle is given by

$$\frac{E^{(1)}}{N} = -\frac{3}{2\pi} (k_F a_B) \operatorname{Ry} \approx -\frac{0.916}{r_s} \operatorname{Ry}.$$
 (7)

[A few hints: Note that $q \neq 0$ in the interaction term and show that therefore

$$\langle \mathrm{FS} | c^{\dagger}_{\boldsymbol{k}+\boldsymbol{q},\sigma} c^{\dagger}_{\boldsymbol{k}'-\boldsymbol{q},\sigma'} c_{\boldsymbol{k}',\sigma'} c_{\boldsymbol{k},\sigma} | \mathrm{FS} \rangle = -\delta_{\boldsymbol{k}',\boldsymbol{k}+\boldsymbol{q}} \delta_{\sigma,\sigma'} \theta(k_F - |\boldsymbol{k}+\boldsymbol{q}|) \theta(k_F - |\boldsymbol{k}|)$$

where $\theta(x)$ is the (Heaviside) step function. Convert the sums over \mathbf{k} and \mathbf{q} to integrals over spherical coordinates. Observe that for a fixed \mathbf{q} the \mathbf{k} -integration amounts to finding the volume of the intersection of two spheres of radius k_F displaced from each other by a vector \mathbf{q} .]

3. Single-particle retarded Green function for noninteracting bosons.

Calculate the Fourier transform of the single-particle retarded Green function, $G_0^R(\nu, \omega)$, for noninteracting bosons with Hamiltonian

$$H_0 = \sum_{\nu} \xi_{\nu} c_{\nu}^{\dagger} c_{\nu}.$$
 (8)

4. The basis invariance of the trace.

Show that the trace of an operator is independent of the basis chosen to evaluate it. [Hint: First define Tr O as the sum of the diagonal elements of O in some particular, but arbitrarily chosen basis. Then do a transformation to an arbitrary different basis and show that Tr O can be rewritten as the sum of the diagonal elements of O in the new basis.]