TFY4210, Quantum theory of many-particle systems, 2013: Tutorial 9

1. Ferromagnetic Heisenberg model with a spin anisotropy revisited.

Consider again the ferromagnetic Heisenberg model with a spin anisotropy studied in Problem 2 in Tutorial 6, where for $D \geq 0$ it was found that the energy gap $\Delta = 2SD$. Thus the system is gapped for $D > 0$ and gapless for $D = 0$. Discuss this property in light of Goldstone's theorem and the symmetries of the model and its ground states.

2. 0th and 1st order perturbation theory for the interacting electron gas.

Consider the 3-dimensional interacting electron gas (more precisely the so-called jellium model introduced in Problem 2 in Tutorial 3) with Hamiltonian

$$H = \sum_{\boldsymbol{k},\sigma} \frac{\hbar^2 k^2}{2m} c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} + \frac{1}{2\Omega} \sum_{\boldsymbol{q}\neq 0} \sum_{\boldsymbol{k},\sigma} \sum_{\boldsymbol{k}',\sigma'} \frac{e^2}{\varepsilon_0 q^2} c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\sigma} c^\dagger_{\boldsymbol{k}'-\boldsymbol{q},\sigma'} c_{\boldsymbol{k}',\sigma'} c_{\boldsymbol{k},\sigma}. \tag{1}$$

It will be convenient to introduce the length scale defined by the Bohr radius $a_B = \frac{4\pi\varepsilon_0\hbar^2}{me^2}$ and the energy scale $\mathrm{Ry} = \frac{\hbar^2}{2ma_B^2}$ (the Rydberg). Let r_0 be the average distance between electrons (defined as the radius of a sphere containing exactly one electron) and define the dimensionless quantity $r_s \equiv r_0/a_B$.

(a) First consider the noninteracting electron gas, whose Hamiltonian is given by the kinetic energy term only. Its ground state is the filled Fermi sphere $|\mathrm{FS}\rangle$ with radius k_F. Show that

$$k_F a_B = \left(\frac{9\pi}{4}\right)^{1/3} \frac{1}{r_s} \tag{2}$$

and that the ground state energy per particle is given by

$$\frac{E^{(0)}}{N} = \frac{3}{5}(k_F a_B)^2\ \mathrm{Ry} \approx \frac{2.211}{r_s^2}\ \mathrm{Ry}. \tag{3}$$

(Here you may make use of results already derived in the lectures for $E^{(0)}/N$ and the relation between k_F and the electron density.)

(b) Next consider the interaction term in (1) as a perturbation on the kinetic energy term. Show that the 1st order correction to the ground state energy per particle is given by

$$\frac{E^{(1)}}{N} = -\frac{3}{2\pi}(k_F a_B)\ \mathrm{Ry} \approx -\frac{0.916}{r_s}\ \mathrm{Ry}. \tag{4}$$

[A few hints: Note that $\boldsymbol{q} \neq 0$ in the interaction term and show that therefore

$$\langle \mathrm{FS}| c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\sigma} c^\dagger_{\boldsymbol{k}'-\boldsymbol{q},\sigma'} c_{\boldsymbol{k}',\sigma'} c_{\boldsymbol{k},\sigma} |\mathrm{FS}\rangle = -\delta_{\boldsymbol{k}',\boldsymbol{k}+\boldsymbol{q}} \delta_{\sigma,\sigma'} \theta(k_F - |\boldsymbol{k}+\boldsymbol{q}|) \theta(k_F - |\boldsymbol{k}|)$$

where $\theta(x)$ is the (Heaviside) step function. Convert the sums over $\boldsymbol{k}$ and $\boldsymbol{q}$ to integrals over spherical coordinates. Observe that for a fixed $\boldsymbol{q}$ the $\boldsymbol{k}$-integration amounts to finding the volume of the intersection of two spheres of radius k_F displaced from each other by a vector $\boldsymbol{q}$.]

3. The basis invariance of the trace.

Show that the trace of an operator is independent of the basis chosen to evaluate it. [Hint: First define Tr O as the sum of the diagonal elements of O in some particular, but arbitrarily chosen basis. Then do a transformation to an arbitrary different basis and show that Tr O can be rewritten as the sum of the diagonal elements of O in the new basis.]