

TFY4210, Quantum theory of many-particle systems, 2014:

Tutorial 9

1. 0th and 1st order perturbation theory for the interacting electron gas.

Consider the 3-dimensional interacting electron gas (more precisely the so-called jellium model introduced in Problem 2 in Tutorial 3) with Hamiltonian

$$H = \sum_{\mathbf{k}, \sigma} \frac{\hbar^2 k^2}{2m} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} + \frac{1}{2\Omega} \sum_{\mathbf{q} \neq 0} \sum_{\mathbf{k}, \sigma} \sum_{\mathbf{k}', \sigma'} \frac{e^2}{\varepsilon_0 q^2} c_{\mathbf{k}+\mathbf{q}, \sigma}^\dagger c_{\mathbf{k}'-\mathbf{q}, \sigma'}^\dagger c_{\mathbf{k}', \sigma'} c_{\mathbf{k}, \sigma}. \quad (1)$$

It will be convenient to introduce the length scale defined by the Bohr radius $a_B = \frac{4\pi\varepsilon_0\hbar^2}{me^2}$ and the energy scale $\text{Ry} = \frac{\hbar^2}{2ma_B^2}$ (the Rydberg). Let r_0 be a measure of the average distance between electrons (defined as the radius of a sphere whose volume equals the volume per electron) and define the dimensionless quantity $r_s \equiv r_0/a_B$.

(a) First consider the noninteracting electron gas, whose Hamiltonian is given by the kinetic energy term only. Its ground state is the filled Fermi sphere |FS⟩ with radius k_F . Show that

$$k_F a_B = \left(\frac{9\pi}{4} \right)^{1/3} \frac{1}{r_s} \quad (2)$$

and that the ground state energy per particle is given by

$$\frac{E^{(0)}}{N} = \frac{3}{5} (k_F a_B)^2 \text{Ry} \approx \frac{2.21}{r_s^2} \text{Ry}. \quad (3)$$

(Here you may make use of results already derived in the lectures for $E^{(0)}/N$ and the relation between k_F and the electron density.)

(b) Next consider the interaction term in (1) as a perturbation on the kinetic energy term. Show that the 1st order correction to the ground state energy per particle is given by

$$\frac{E^{(1)}}{N} = -\frac{3}{2\pi} (k_F a_B) \text{Ry} \approx -\frac{0.916}{r_s} \text{Ry}. \quad (4)$$

[A few hints: Note that $\mathbf{q} \neq 0$ in the interaction term and show that therefore

$$\langle \text{FS} | c_{\mathbf{k}+\mathbf{q},\sigma}^\dagger c_{\mathbf{k}'-\mathbf{q},\sigma'}^\dagger c_{\mathbf{k}',\sigma'} c_{\mathbf{k},\sigma} | \text{FS} \rangle = -\delta_{\mathbf{k}',\mathbf{k}+\mathbf{q}} \delta_{\sigma,\sigma'} \theta(k_F - |\mathbf{k} + \mathbf{q}|) \theta(k_F - |\mathbf{k}|)$$

where $\theta(x)$ is the (Heaviside) step function. Convert the sums over \mathbf{k} and \mathbf{q} to integrals over spherical coordinates. Observe that for a fixed \mathbf{q} the \mathbf{k} -integration amounts to finding the volume of the intersection of two spheres of radius k_F displaced from each other by a vector \mathbf{q} .]

2. Single-particle retarded Green function for noninteracting bosons.

Calculate the Fourier transform of the single-particle retarded Green function, $G_0^R(\nu, \omega)$, for noninteracting bosons with Hamiltonian

$$H_0 = \sum_{\nu} \xi_{\nu} c_{\nu}^\dagger c_{\nu}. \quad (5)$$

3. The basis invariance of the trace.

Show that the trace of an operator is independent of the basis chosen to evaluate it. [Hint: First define $\text{Tr } O$ as the sum of the diagonal elements of O in some particular, but arbitrarily chosen basis. Then do a transformation to an arbitrary different basis and show that $\text{Tr } O$ can be rewritten as the sum of the diagonal elements of O in the new basis.]