TFY4210, Quantum theory of many-particle systems, 2015: Tutorial 9

In this tutorial set, all problems concern fermions.

1. Relationship between $G^<(\nu, \omega)$ and the spectral function $A(\nu, \omega)$.

(a) Starting from the definition of the "lesser" Green's function, show that it can be written

$$G^<(\nu; t, t') = \frac{i}{Z}\sum_{n,m} e^{-\beta E_m} e^{i(E_n - E_m)(t-t')} |\langle m|c_\nu^\dagger|n\rangle|^2. \tag{1}$$

and that its Fourier transform is

$$G^<(\nu, \omega) = \frac{2\pi i}{Z}\sum_{n,m} e^{-\beta E_m} |\langle m|c_\nu^\dagger|n\rangle|^2 \delta(\omega + E_n - E_m). \tag{2}$$

(b) Starting from Eq. (53) in the lecture notes, show that $A(\nu, \omega)$ can be rewritten as

$$A(\nu, \omega) = (1 + e^{\beta\omega})\frac{1}{Z}\sum_{n,m} |\langle m|c_\nu^\dagger|n\rangle|^2 e^{-\beta E_m} \delta(\omega + E_n - E_m). \tag{3}$$

(c) Show that

$$-iG^<(\nu, \omega) = 2\pi A(\nu, \omega) n_F(\omega) \tag{4}$$

which is Eq. (56) in the lecture notes. (Eq. (55) can be proved in a similar way.)

2. An alternative form of the Lehmann representation.

Show that [Eq. (54) in the lecture notes]

$$G^R(\nu, \omega) = \int_{-\infty}^{\infty} d\omega' \, \frac{A(\nu, \omega')}{\omega - \omega' + i\eta}. \tag{5}$$

3. Calculating $G^R(\nu, t)$ from $G^R(\nu, \omega)$ by contour integration.

In the lectures we calculated $G^R(\nu, \omega)$ from a knowledge of $G^R(\nu, t)$. But suppose that you instead know $G^R(\nu, \omega)$ and want to calculate $G^R(\nu, t)$ from it. Starting from the Lehmann representation for $G^R(\nu, \omega)$ [Eq. (50) in the lecture notes], calculate

$$G^R(\nu, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega t} G^R(\nu, \omega) \tag{6}$$

by using **contour integration**: Consider the cases $t > 0$ and $t < 0$ separately. For each case, identify which half-plane the contour must be closed in. By using the residue theorem to evaluate the resulting contour integrals, show that

$$G^R(\nu, t) = -i\theta(t)\frac{1}{Z}\sum_{n,m}\left(e^{-\beta E_n} + e^{-\beta E_m}\right) e^{i(E_n - E_m)t}|\langle m|c_\nu^\dagger|n\rangle|^2 \tag{7}$$

which is Eq. (48) in the lecture notes.

4. Fermi liquids.

Consider the following simplified model for the spectral function of a Fermi liquid for wavevectors $\boldsymbol{k}$ in the vicinity of the Fermi surface:

$$A(\boldsymbol{k}\sigma, \omega) = Z\delta(\omega - \xi_{\boldsymbol{k}}) + (1 - Z)\,\frac{\theta(W - |\omega|)}{2W}. \tag{8}$$

Here Z and W are constants (real and positive, with $Z \le 1$), and $\theta(x)$ is the Heaviside (step) function.

(a) Briefly compare and contrast this expression with the spectral function of a Fermi liquid as discussed in the lecture notes [Eq. (60)].

(b) Show that (8) satisfies the sum rule

$$\int_{-\infty}^{\infty} d\omega\, A(\boldsymbol{k}\sigma, \omega) = 1. \tag{9}$$

(c) Show that at zero temperature the momentum distribution function $\langle c_{\boldsymbol{k}\sigma}^\dagger c_{\boldsymbol{k}\sigma}\rangle$ of the Fermi liquid described by (8) has a jump of magnitude Z as $k = |\boldsymbol{k}|$ crosses the Fermi surface.