

TFY4210, Quantum theory of many-particle systems, 2016:

Tutorial 9

In this tutorial set, all problems concern fermions.

1. Relationship between $G^<(\nu, \omega)$ and the spectral function $A(\nu, \omega)$.

(a) Starting from the definition of the "lesser" Green's function, show that it can be written

$$G^<(\nu; t, t') = \frac{i}{Z} \sum_{n,m} e^{-\beta E_m} e^{i(E_n - E_m)(t - t')} |\langle m | c_\nu^\dagger | n \rangle|^2. \quad (1)$$

and that its Fourier transform is

$$G^<(\nu, \omega) = \frac{2\pi i}{Z} \sum_{n,m} e^{-\beta E_m} |\langle m | c_\nu^\dagger | n \rangle|^2 \delta(\omega + E_n - E_m). \quad (2)$$

(b) Starting from Eq. (53) in the lecture notes, show that $A(\nu, \omega)$ can be rewritten as

$$A(\nu, \omega) = (1 + e^{\beta\omega}) \frac{1}{Z} \sum_{n,m} |\langle m | c_\nu^\dagger | n \rangle|^2 e^{-\beta E_m} \delta(\omega + E_n - E_m). \quad (3)$$

(c) Show that

$$-iG^<(\nu, \omega) = 2\pi A(\nu, \omega) n_F(\omega) \quad (4)$$

which is Eq. (56) in the lecture notes. (Eq. (55) can be proved in a similar way.)

2. An alternative form of the Lehmann representation.

Show that [Eq. (54) in the lecture notes]

$$G^R(\nu, \omega) = \int_{-\infty}^{\infty} d\omega' \frac{A(\nu, \omega')}{\omega - \omega' + i\eta}. \quad (5)$$

3. Calculating $G^R(\nu, t)$ from $G^R(\nu, \omega)$ by contour integration.

In the lectures we calculated $G^R(\nu, \omega)$ from a knowledge of $G^R(\nu, t)$. But suppose that you instead know $G^R(\nu, \omega)$ and want to calculate $G^R(\nu, t)$ from it. Starting from the Lehmann representation for $G^R(\nu, \omega)$ [Eq. (50) in the lecture notes], calculate

$$G^R(\nu, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} G^R(\nu, \omega) \quad (6)$$

by using **contour integration**: Consider the cases $t > 0$ and $t < 0$ separately. For each case, identify which half-plane the contour must be closed in. By using the residue theorem to evaluate the resulting contour integrals, show that

$$G^R(\nu, t) = -i\theta(t)\frac{1}{Z} \sum_{n,m} (e^{-\beta E_n} + e^{-\beta E_m}) e^{i(E_n - E_m)t} |\langle m | c_\nu^\dagger | n \rangle|^2 \quad (7)$$

which is Eq. (48) in the lecture notes.

4. Fermi liquids.

Consider the following simplified model for the spectral function of a Fermi liquid for wavevectors \mathbf{k} in the vicinity of the Fermi surface:

$$A(\mathbf{k}\sigma, \omega) = Z\delta(\omega - \xi_{\mathbf{k}}) + (1 - Z) \frac{\theta(W - |\omega|)}{2W}. \quad (8)$$

Here Z and W are constants (real and positive, with $Z \leq 1$), and $\theta(x)$ is the Heaviside (step) function.

(a) Briefly compare and contrast this expression with the spectral function of a Fermi liquid as discussed in the lecture notes [Eq. (60)].

(b) Show that (8) satisfies the sum rule

$$\int_{-\infty}^{\infty} d\omega A(\mathbf{k}\sigma, \omega) = 1. \quad (9)$$

(c) Show that at zero temperature the momentum distribution function $\langle c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} \rangle$ of the Fermi liquid described by (8) has a jump of magnitude Z as $k = |\mathbf{k}|$ crosses the Fermi surface.