TFY4210, Quantum theory of many-particle systems, 2016: Solution to tutorial 10

A model of interacting spins on a one-dimensional lattice.

Note that in this solution I generally do not write hats (ˆ) on operators. Also, in this solution I have included an overall multiplicative parameter $J > 0$ of dimension energy in the expression for the Hamiltonian (this factor was missing in the problem text). Thus the original Hamiltonian is written

$$H = -J \sum_j \left[\frac{1+\gamma}{2} \sigma_j^x \sigma_{j+1}^x + \frac{1-\gamma}{2} \sigma_j^y \sigma_{j+1}^y + \lambda \sigma_j^z \right]. \quad (1)$$

(a) We have

$$[\sigma_i^+, \sigma_j^-] = \left[\prod_{m=1}^{i-1} (1-2n_m) \right] c_i \left[\prod_{n=1}^{j-1} (1-2n_n) \right] c_j^\dagger - \left[\prod_{n=1}^{j-1} (1-2n_n) \right] c_j^\dagger \left[\prod_{m=1}^{i-1} (1-2n_m) \right] c_i. \quad (2)$$

Taking $i < j$, noting that $(1-2n_i)$ is the only factor involving a number operator that doesn't commute with everything else, the expression can be simplified to

$$\begin{aligned} [\sigma_i^+, \sigma_j^-] &= \left[\prod_{m=1}^{i-1} (1-2n_m) \right] \left[\prod_{n=1}^{i-1} (1-2n_n) \right] \left[\prod_{n=i+1}^{j-1} (1-2n_n) \right] \left[c_i (1-2n_i) c_j^\dagger - (1-2n_i) c_j^\dagger c_i \right] \\ &= \left[\prod_{m=1}^{i-1} (1-2n_m) \right] \left[\prod_{n=1}^{i-1} (1-2n_n) \right] \left[\prod_{n=i+1}^{j-1} (1-2n_n) \right] \left[c_i (1-2n_i) + (1-2n_i) c_i \right] c_j^\dagger. \end{aligned} \quad (3)$$

The rightmost expression inside square brackets can be rewritten as

$$c_i - 2c_i n_i + c_i - 2n_i c_i = 2c_i + 2(-c_i - n_i c_i) - 2n_i c_i = -4n_i c_i = -4c_i^\dagger c_i c_i = 0 \quad (4)$$

(here we used $[n_i, c_i] = -c_i$ and $c_i^2 = 0$). Thus $[\sigma_i^+, \sigma_j^-] = 0$. QED.

(b) We have

$$\begin{aligned} \sigma_i^+ \sigma_{i+1}^+ &= \left[\prod_{j<i} (1-2n_j) \right] c_i \left[\prod_{m<i+1} (1-2n_m) \right] c_{i+1} \\ &= \left[\prod_{j<i} (1-2n_j) \right] c_i \left[\prod_{m<i} (1-2n_m) \right] (1-2n_i) c_{i+1} \\ &= \left[\prod_{j<i} (1-2n_j)^2 \right] c_i (1-2n_i) c_{i+1}. \end{aligned} \quad (5)$$

The expression inside square brackets is equal to 1. This can be seen by direct calculation:[1]

$$(1-2n_j)^2 = 1-4n_j+4n_j^2 = 1-4n_j+4n_j = 1 \tag{6}$$

where we used $n_j^2 = n_j$. Furthermore,

$$c_i(1-2n_i) = c_i - 2c_i n_i = c_i + 2(-c_i - n_i c_i) = -c_i - 2n_i c_i = -c_i \tag{7}$$

where we again used $c_i^2 = 0$.[2] Thus

$$\sigma_i^+ \sigma_{i+1}^+ = -c_i c_{i+1} = c_{i+1} c_i. \tag{8}$$

(c) Starting from the fermionic Hamiltonian

$$H = -J \sum_i \left[(c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1}) + \gamma(c_{i+1} c_i + c_i^\dagger c_{i+1}^\dagger) - 2\lambda c_i^\dagger c_i + \lambda \right], \tag{9}$$

inserting the Fourier expansion of the cre/ann operators gives

$$H = -NJ\lambda - J \sum_{k,k'} \Big\{ c_k^\dagger c_{k'} (e^{-ika} + e^{ik'a}) F(-k,k') \tag{10}$$

$$+ \gamma[c_k c_{k'} e^{ika} F(k,k') + c_k^\dagger c_{k'}^\dagger e^{-ik'a} F(-k,-k')] - 2\lambda c_k^\dagger c_{k'} F(-k,k') \Big\}, \tag{11}$$

where $F(k,k') = N^{-1} \sum_{j=1}^{N} e^{i(k+k')ja}$. Using $F(k,k') = \delta_{k,-k'}$ (Eq. (31) in problem text) gives

$$H = -J \sum_k \Big\{ 2(\cos ka - \lambda) c_k^\dagger c_k + \gamma(c_k c_{-k} + c_k^\dagger c_{-k}^\dagger) e^{ika} + \lambda \Big\}. \tag{12}$$

The anomalous part can be rewritten as

$$\sum_k (c_k c_{-k} + c_k^\dagger c_{-k}^\dagger) e^{ika} = \sum_k (c_{-k} c_k + c_{-k}^\dagger c_k^\dagger) e^{-ika} = -\sum_k (c_k c_{-k} + c_k^\dagger c_{-k}^\dagger) e^{-ika}. \tag{13}$$

To get the first equality, define $k' = -k$ so that the sum becomes a sum over the values of $k' = -k$ (as the allowed values of k are symmetrically distributed around 0, the sum over $k' = -k$ goes over the same values as the sum over k), and then just rename $k' \to k$ (can be done since it is a dummy summation variable). To get the second equality we used $\{c_k, c_{-k}\} = \{c_k^\dagger, c_{-k}^\dagger\} = 0$. Writing the anomalous part as the average of the first and last expressions in (13) gives

$$\frac{1}{2} \sum_k (c_k c_{-k} + c_k^\dagger c_{-k}^\dagger)(e^{ika} - e^{-ika}) = i \sum_k \sin ka (c_k c_{-k} + c_k^\dagger c_{-k}^\dagger). \tag{14}$$

[1] Alternatively, the same conclusion can be reached from the fact that the eigenvalues of n_j are restricted to 0 and 1, so the action of $(1-2n_j)^2$ on a basis state with a definite number of fermions on site j just gives a multiplicative factor $(1 - 2\cdot(0 \text{ or } 1))^2 = (\pm 1)^2 = 1$.

[2] Alternatively, the same conclusion can be reached by noting that for the operator $c_i(1-2n_i)$ to give a nonzero result, the site i must be occupied before c_i acts, and thus $1-2n_i$ acting before will give a factor $1 - 2\cdot 1 = -1$ in that case. (If the site is empty, the factor would instead be $+1$, but in that case c_i will in any case kill the state so the sign is insignificant.)

Putting everything together, we find that

$$H = J \sum_k [2(\lambda - \cos ka) c_k^\dagger c_k + i\gamma \sin ka (c_{-k} c_k + c_{-k}^\dagger c_k^\dagger) - \lambda]. \tag{15}$$

You can check that this expression is Hermitian, as it should be.

(d) We start from

$$d_k = u_k c_k - i v_k c_{-k}^\dagger. \tag{16}$$

Taking the adjoint and using that u_k and v_k are real gives $d_k^\dagger = u_k c_k^\dagger + i v_k c_{-k}$. Thus $d_{-k}^\dagger = u_{-k} c_{-k}^\dagger + i v_{-k} c_k$. Using the (anti-)symmetries of u_k and v_k this becomes

$$d_{-k}^\dagger = u_k c_{-k}^\dagger - i v_k c_k. \tag{17}$$

We now have two equations ((16) and (17)) in two unknowns c_k and $c_{-k}^\dagger$. Multiplying (16) by u_k and (17) by iv_k, and adding the resulting equations, the contribution from $c_{-k}^\dagger$ cancels, leaving

$$u_k d_k + i v_k d_{-k}^\dagger = (u_k^2 + v_k^2) c_k = c_k, \tag{18}$$

which was to be shown. In the last equality here we used that

$$u_k^2 + v_k^2 = 1, \tag{19}$$

which is a direct consequence of the relations $u_k = \cos(\theta_k/2)$, $v_k = \sin(\theta_k/2)$.

(e) We can use (18) to rewrite (15) in terms of the d-operators. The anomalous terms are (you should show this)

$$iJ \sum_k [(\lambda - \cos ka) 2 u_k v_k - \gamma \sin ka (u_k^2 - v_k^2)] d_k^\dagger d_{-k}^\dagger + \text{ h.c.} \tag{20}$$

Using

$$\begin{aligned} u_k^2 - v_k^2 &= \cos\theta_k, & (21)\\ 2u_k v_k &= \sin\theta_k, & (22) \end{aligned}$$

and choosing θ_k such that the anomalous part vanishes, we get the condition

$$(\lambda - \cos ka)\sin\theta_k = \gamma \sin ka \cos\theta_k, \tag{23}$$

i.e.

$$\tan\theta_k = \frac{\gamma \sin ka}{\lambda - \cos ka}. \tag{24}$$

(Note that the rhs is odd in k, consistent with the property $\theta_{-k} = -\theta_k$.)

(f) For θ_k satisfying (24) the Hamiltonian reduces to the ordinary (i.e., non-anomalous) part which is found to be

$$\begin{aligned} H &= J\sum_k\{[2(\lambda-\cos ka)(u_k^2-v_k^2)+2\gamma\sin ka\cdot 2u_kv_k]d_k^\dagger d_k+2(\lambda-\cos ka)v_k^2 \\ &- \gamma\sin ka\cdot 2u_kv_k-\lambda\} \end{aligned} \tag{25}$$

which is in the desired form

$$H=\sum_k \varepsilon_k d_k^\dagger d_k + C. \tag{26}$$

Using (21)-(22) and

$$v_k^2=\frac{1}{2}(1-\cos\theta_k) \tag{27}$$

(which follows from subtracting (21) from (19)), we find

$$\begin{aligned} \varepsilon_k &= 2J[(\lambda-\cos ka)\cos\theta_k+\gamma\sin ka\sin\theta_k], && (28)\\ C &= J\sum_k[(\lambda-\cos ka)(1-\cos\theta_k)-\gamma\sin ka\sin\theta_k-\lambda]. && (29) \end{aligned}$$

The constant C can be rewritten as

$$C=-J\sum_k[(\lambda-\cos ka)\cos\theta_k+\gamma\sin ka\sin\theta_k+\cos ka]=-\frac{1}{2}\sum_k\varepsilon_k, \tag{30}$$

where we used $\sum_k \cos ka=0$ to get the last equality. We can furthermore rewrite ε_k as

$$\begin{aligned} \varepsilon_k &= 2J\cos\theta_k[\lambda-\cos ka+\gamma\sin ka\tan\theta_k]=2J\cos\theta_k\left[\lambda-\cos ka+\gamma\sin ka\frac{\gamma\sin ka}{\lambda-\cos ka}\right] \\ &= \frac{2J\cos\theta_k}{\lambda-\cos ka}[(\lambda-\cos ka)^2+\gamma^2\sin^2 ka]. \end{aligned} \tag{31}$$

From the identity $\cos^2 x=(1+\tan^2 x)^{-1}$ we have

$$\cos^2\theta_k=\frac{1}{1+\tan^2\theta_k}=\frac{(\lambda-\cos ka)^2}{(\lambda-\cos ka)^2+\gamma^2\sin^2 ka}. \tag{32}$$

This only determines $\cos\theta_k$ up to a sign. The condition (24) does not put any constraints on this sign. Here we will choose the sign such that[3]

$$\cos\theta_k=\frac{\lambda-\cos ka}{\sqrt{(\lambda-\cos ka)^2+\gamma^2\sin^2 ka}}. \tag{33}$$

(This choice satisfies $\cos\theta_{-k}=\cos\theta_k$ which follows from $\theta_{-k}=-\theta_k$.) Inserting this into (31) gives

$$\varepsilon_k=2J\sqrt{(\lambda-\cos ka)^2+\gamma^2\sin^2 ka}. \tag{34}$$

[3] Alternative choices for this sign are possible. For example, one could replace the rhs of (33) by its absolute value. Different choices will only lead to differences in the mathematical description of the model with no consequences for the physics.

Summarizing, the Hamiltonian is

$$H = \sum_k \varepsilon_k \left(d_k^\dagger d_k - \frac{1}{2} \right) \tag{35}$$

where ε_k is given by (34). Thus we see that, remarkably, the particular model (1) of *interacting* spins on a one-dimensional lattice can be rewritten as (35) describing *noninteracting* fermions.

(g) From (34) one sees that $\varepsilon_k \geq 0$ for all k, and therefore all occupation numbers for the d-fermions are zero in the ground state. Thus the ground state energy is

$$E_0(\gamma, \lambda) = \sum_k \varepsilon_k \left(0 - \frac{1}{2} \right) = -J \sum_k \sqrt{(\lambda - \cos ka)^2 + \gamma^2 \sin^2 ka}. \tag{36}$$

(h) Since the ground state $|G\rangle$ contains no d-fermions, it is the vacuum of the d-fermions:

$$d_k |G\rangle = 0 \quad \text{for all } k. \tag{37}$$

These equations define $|G\rangle$. We note that d_k only involves c_k and $c_{-k}^\dagger$ and that d_{-k} only involves c_{-k} and $c_k^\dagger$. This means that the problem of finding the ground state decouples into independent subproblems, one for each $|k|$, so the ground state can be written $|G\rangle = \prod_{k \geq 0} |G_k\rangle$, where $|G_k\rangle$ is the vacuum of d_k and d_{-k}. For concreteness let $k > 0$ in the following. Then k and $-k$ are inequivalent wavevectors representing different states. Because each of them can be occupied by either 0 or 1 c-fermions, $|G_k\rangle$ can be expanded as

$$|G_k\rangle = \alpha_k |0\rangle + \beta_k c_k^\dagger |0\rangle + \gamma_k c_{-k}^\dagger |0\rangle + \delta_k c_k^\dagger c_{-k}^\dagger |0\rangle \tag{38}$$

where $|0\rangle$ is the vacuum of the c-fermions and α_k, β_k, γ_k and δ_k are parameters to be determined from the two conditions

$$\begin{aligned} 0 &= d_k |G_k\rangle = (\cos \frac{\theta_k}{2} c_k - i \sin \frac{\theta_k}{2} c_{-k}^\dagger) |G_k\rangle, & (39) \\ 0 &= d_{-k} |G_k\rangle = (\cos \frac{\theta_k}{2} c_{-k} + i \sin \frac{\theta_k}{2} c_k^\dagger) |G_k\rangle. & (40) \end{aligned}$$

The condition (39) becomes

$$\begin{aligned} 0 &= \cos \frac{\theta_k}{2} \beta_k c_k c_k^\dagger |0\rangle + \cos \frac{\theta_k}{2} \delta_k c_k c_k^\dagger c_{-k}^\dagger |0\rangle - i \sin \frac{\theta_k}{2} \alpha_k c_{-k}^\dagger |0\rangle - i \sin \frac{\theta_k}{2} \beta_k c_{-k}^\dagger c_k^\dagger |0\rangle \\ &= \cos \frac{\theta_k}{2} \beta_k |0\rangle + \cos \frac{\theta_k}{2} \delta_k c_{-k}^\dagger |0\rangle - i \sin \frac{\theta_k}{2} \alpha_k c_{-k}^\dagger |0\rangle + i \sin \frac{\theta_k}{2} \beta_k c_k^\dagger c_{-k}^\dagger |0\rangle, \qquad (41) \end{aligned}$$

which gives

$$\begin{aligned} \beta_k &= 0, & (42) \\ \cos \frac{\theta_k}{2} \delta_k - i \sin \frac{\theta_k}{2} \alpha_k &= 0. & (43) \end{aligned}$$

The condition (40) becomes

$$
\begin{aligned}
0 &= \cos\frac{\theta_k}{2}\gamma_k c_{-k}c_{-k}^\dagger|0\rangle + \cos\frac{\theta_k}{2}\delta_k c_{-k}c_k^\dagger c_{-k}^\dagger|0\rangle + i\sin\frac{\theta_k}{2}\alpha_k c_k^\dagger|0\rangle + i\sin\frac{\theta_k}{2}\gamma_k c_k^\dagger c_{-k}^\dagger|0\rangle \\
&= \cos\frac{\theta_k}{2}\gamma_k|0\rangle - \cos\frac{\theta_k}{2}\delta_k c_k^\dagger|0\rangle + i\sin\frac{\theta_k}{2}\alpha_k c_k^\dagger|0\rangle + i\sin\frac{\theta_k}{2}\gamma_k c_k^\dagger c_{-k}^\dagger|0\rangle, \qquad (44)
\end{aligned}
$$

which gives

$$
\begin{aligned}
\gamma_k &= 0, \qquad (45)\\
-\cos\frac{\theta_k}{2}\delta_k + i\sin\frac{\theta_k}{2}\alpha_k &= 0. \qquad (46)
\end{aligned}
$$

The last equality here is identical to what we got from the first condition. Thus we find $\beta_k = \gamma_k = 0$, and

$$
\begin{aligned}
\alpha_k &= \mathcal{N}\cos\frac{\theta_k}{2}, \qquad (47)\\
\delta_k &= i\mathcal{N}\sin\frac{\theta_k}{2}, \qquad (48)
\end{aligned}
$$

where $\mathcal{N}$ is a normalization constant. Normalization of (38) requires $|\alpha_k|^2 + |\beta_k|^2 + |\gamma_k|^2 + |\delta_k|^2 = 1$, which implies $|\mathcal{N}|^2 = 1$, so we can take $\mathcal{N} = 1$. Thus for $k > 0$ one finds

$$
|G_k\rangle = \left(\cos\frac{\theta_k}{2} + i\sin\frac{\theta_k}{2}c_k^\dagger c_{-k}^\dagger\right)|0\rangle. \qquad (49)
$$

In contrast, for $k = 0$ we have $k = -k$ and thus only a single wavevector is involved. In this case one instead finds (can you show this?)

$$
|G_0\rangle = \begin{cases} c_0^\dagger|0\rangle & \text{if } \lambda < 1, \\ |0\rangle & \text{if } \lambda > 1. \end{cases} \qquad (50)
$$

Summarizing, we can thus write

$$
|G\rangle = \left(\prod_{k\geq 0}\hat{G}_k\right)|0\rangle \quad \text{where } \hat{G}_k = \begin{cases} \cos(\theta_k/2) + i\sin(\theta_k/2)c_k^\dagger c_{-k}^\dagger & \text{for } k > 0, \\ \Theta(\lambda-1) + \Theta(1-\lambda)c_0^\dagger & \text{for } k = 0, \end{cases} \qquad (51)
$$

where $\Theta(x)$ is the Heaviside (step) function.

It is of course of interest to understand the expression (51) for the ground state $|G\rangle$ in terms of the behaviour of the spins in the original spin model. To illustrate this, and as a check of our results, let us consider the special case $\lambda \to \infty$ which is easy to analyze. In this limit H is dominated by the term $-J\lambda\sum_j \sigma_j^z$, so all spins will point up in the ground state, which thus is a product over all sites j of the eigenstate of the operator σ_j^z with eigenvalue $+1$. The Jordan-Wigner expression $\sigma_j^z = 1 - 2n_j$ shows that this implies the eigenvalue 0 for all fermion number operators n_j. Thus the ground state should correspond to the vacuum state $|0\rangle$ of the c-fermions. Let us check this: For $\lambda \to \infty$, one sees from (24) and (33) that

$\theta_k \to 0$. We then see from (51) that in the limit $\lambda \to \infty$, $\hat{G}_k$ becomes 1 for each k in the product, so the ground state is indeed given by $|G\rangle = |0\rangle$.

(i) The diagonal form (35) of H implies that excited states are obtained by creating d-fermions. As such a fermion has energy ε_k, where k is its wavevector, the lowest excitation energy is obtained by creating only one d-fermion, and choosing its wavevector k such that ε_k is minimized. We are here looking for the parameter values (γ, λ) for which the system has *gapless excitations*, i.e. the lowest excitation energy is 0, or approaches 0 in the thermodynamic limit $N \to \infty$. So we should look for solutions of the equation $\varepsilon_k = 0$ in this limit. The expression inside the square root in (34) must thus vanish, i.e. $(\lambda - \cos ka)^2 + \gamma^2 \sin^2 ka = 0$. Since both terms are nonnegative it follows that we must have

$$\lambda = \cos ka \quad \text{and} \quad \gamma \sin ka = 0. \tag{52}$$

One solution is $k = 0$ and $\lambda = 1$, with γ arbitrary. So the model is gapless along the line γ arbitrary, $\lambda = 1$ in (γ, λ)-space. Another solution is $\gamma = 0$ and $\lambda = \cos ka$, where the latter equation can be satisfied for some k provided that $-1 \le \lambda \le 1$. So the model is also gapless along the line $\gamma = 0$, $-1 \le \lambda \le 1$ in (γ, λ)-space.