TFY4210, Quantum theory of many-particle systems, 2016: Solution to tutorial 11

1. Matsubara and retarded Green functions for noninteracting bosons.

(a) We have

$$\mathcal{G}^{(0)}(\nu,\tau) = -\langle T_\tau (c_\nu(\tau) c_\nu^\dagger(0))\rangle. \tag{1}$$

Here

$$c_\nu(\tau) = e^{H_0\tau} c_\nu e^{-H_0\tau}. \tag{2}$$

One way to calculate this operator is to solve the differential equation obeyed by this operator. Differentiating wrt t gives

$$\frac{dc_\nu(\tau)}{d\tau} = e^{H_0\tau}H_0 c_\nu e^{-H_0\tau} + e^{H_0\tau}c_\nu(-H_0)e^{-H_0\tau} = e^{H_0\tau}\underbrace{[H_0, c_\nu]}_{-\xi_\nu c_\nu} e^{-H_0\tau} = -\xi_\nu c_\nu(\tau). \tag{3}$$

The solution to this differential equation, for the initial condition $c_\nu(0) = c_\nu$, is

$$c_\nu(\tau) = e^{-\xi_\nu \tau} c_\nu. \tag{4}$$

Another way is to use the Baker-Hausdorff formula (we omit the details here). Yet another way is to note that (2) can be simplified to (can you see why?)

$$c_\nu(\tau) = e^{\tau\xi_\nu \hat{n}_\nu} c_\nu e^{-\tau\xi_\nu\hat{n}_\nu}. \tag{5}$$

If we now act with this operator on a many-particle basis state with n_ν bosons in single-particle state ν, we get (denoting the basis state as $|n_\nu\rangle$ for short)

$$\begin{aligned} c_\nu(\tau)|n_\nu\rangle &= e^{\tau\xi_\nu\hat{n}_\nu} c_\nu e^{-\tau\xi_\nu\hat{n}_\nu}|n_\nu\rangle \\ &= e^{-\tau\xi_\nu n_\nu} e^{\tau\xi_\nu\hat{n}_\nu} c_\nu|n_\nu\rangle \\ &= e^{-\tau\xi_\nu n_\nu} e^{\tau\xi_\nu(n_\nu-1)} c_\nu|n_\nu\rangle \\ &= e^{-\tau\xi_\nu} c_\nu|n_\nu\rangle. \end{aligned} \tag{6}$$

(A fine point: For the case $n_\nu = 0$ the factor $e^{\tau\xi_\nu(n_\nu-1)}$ wouldn't actually appear, but the expression is correct also for this case since $c_\nu|n_\nu\rangle = 0$ then.) As this expression is valid for any basis state $|n_\nu\rangle$ we arrive at (4) as an operator identity.

Let us return to (1). For $\tau > 0$ the operators are already time-ordered, so

$$\begin{aligned} \mathcal{G}^{(0)}(\nu,\tau>0) &= -\langle c_\nu(\tau)c_\nu^\dagger(0)\rangle = -e^{-\xi_\nu\tau}\langle c_\nu c_\nu^\dagger\rangle \\ &= -e^{-\xi_\nu\tau}(1+\langle c_\nu^\dagger c_\nu\rangle) = -e^{-\xi_\nu\tau}(1+n_B(\xi_\nu)). \end{aligned} \tag{7}$$

For $\tau < 0$ the time ordering operator reverses the order of the operators. As they are bosonic, there is no sign change associated with this time ordering. Thus

$$\mathcal{G}^{(0)}(\nu,\tau<0) = -\langle c_\nu^\dagger(0)c_\nu(\tau)\rangle = -e^{-\xi_\nu\tau}\langle c_\nu^\dagger c_\nu\rangle = -e^{-\xi_\nu\tau} n_B(\xi_\nu). \tag{8}$$

Thus we can write

$$\mathcal{G}^{(0)}(\nu,\tau) = -e^{-\xi_\nu \tau}\left[\theta(\tau)(1+n_B(\xi_\nu)) + \theta(-\tau)n_B(\xi_\nu)\right]. \tag{9}$$

(b) We have

$$\begin{aligned}
\mathcal{G}^{(0)}(\nu, i\omega_n) &= \int_0^\beta d\tau\, e^{i\omega_n\tau}\mathcal{G}^{(0)}(\nu,\tau) = -(1+n_B(\xi_\nu))\int_0^\beta d\tau\, e^{(i\omega_n-\xi_\nu)\tau} \\
&= -(1+n_B(\xi_\nu))\frac{1}{i\omega_n-\xi_\nu}(e^{(i\omega_n-\xi_\nu)\beta}-1)
\end{aligned} \tag{10}$$

Now use that for bosonic Matsubara frequencies,

$$\omega_n = \frac{2\pi n}{\beta} \quad \Rightarrow \quad e^{i\omega_n\beta} = 1. \tag{11}$$

Furthermore,

$$1 + n_B(\xi_\nu) = 1 + \frac{1}{e^{\beta\xi_\nu}-1} = \frac{1}{1-e^{-\beta\xi_\nu}}. \tag{12}$$

Inserting these results, the Matsubara Green function becomes

$$\mathcal{G}^{(0)}(\nu, i\omega_n) = \frac{1}{i\omega_n - \xi_\nu}. \tag{13}$$

The retarded Green function is

$$G^{R(0)}(\nu,\omega) = \mathcal{G}^{(0)}(\nu, i\omega_n)\Big|_{i\omega_n\to\omega+i\eta} = \frac{1}{\omega-\xi_\nu+i\eta}. \tag{14}$$

2. Impurity scattering: Impurity average and Feynman diagrams.

(a) For $n=3$ we must consider

$$\begin{aligned}
\overline{\rho(\boldsymbol{k}-\boldsymbol{k}_1)\rho(\boldsymbol{k}_1-\boldsymbol{k}_2)\rho(\boldsymbol{k}_2-\boldsymbol{k}')} &= \sum_{j_1}\sum_{j_2}\sum_{j_3}\left[\prod_i\left(\frac{1}{\Omega}\int d^3R_i\right)\right] \\
&\quad e^{-i(\boldsymbol{k}-\boldsymbol{k}_1)\cdot\boldsymbol{R}_{j_1}}e^{-i(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{R}_{j_2}}e^{-i(\boldsymbol{k}_2-\boldsymbol{k}')\cdot\boldsymbol{R}_{j_3}}.
\end{aligned} \tag{15}$$

We want the contribution corresponding to $j_1 = j_2 \neq j_3$. Then the product of exponentials simplifies to

$$e^{-i(\boldsymbol{k}-\boldsymbol{k}_2)\cdot\boldsymbol{R}_{j_1}}e^{-i(\boldsymbol{k}_2-\boldsymbol{k}')\cdot\boldsymbol{R}_{j_3}} \tag{16}$$

Thus the average over R_{j_1} gives $\delta_{\boldsymbol{k},\boldsymbol{k}_2}$, while the average over R_{j_3} gives $\delta_{\boldsymbol{k}_2,\boldsymbol{k}'}$. The other $N-2$ averages just give 1. Thus the total average is $\delta_{\boldsymbol{k},\boldsymbol{k}_2}\delta_{\boldsymbol{k}_2,\boldsymbol{k}'}$ which can be rewritten $\delta_{\boldsymbol{k},\boldsymbol{k}_2}\delta_{\boldsymbol{k},\boldsymbol{k}'}$. The site summations for this contribution give

$$\sum_{j_1}\sum_{j_2}\sum_{j_3}\delta_{j_1,j_2}(1-\delta_{j_1,j_3}) = N^2 - N \approx N^2. \tag{17}$$

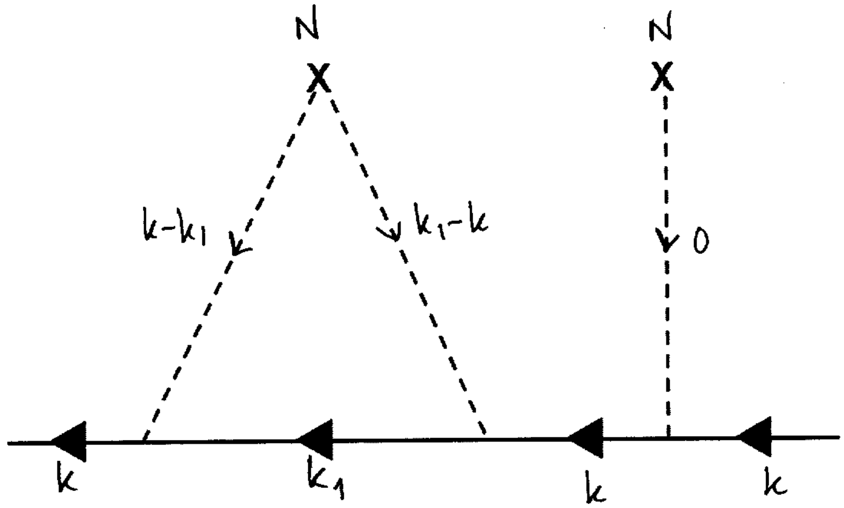


Figure 1: The Feynman diagram for problem 2(a).

Thus the contribution to the impurity average is $N^2\delta_{\boldsymbol{k},\boldsymbol{k}_2}\delta_{\boldsymbol{k},\boldsymbol{k}'}$. Considering the impurity average of Eq. (120) in the notes, the contribution to $\overline{\mathcal{G}^{(3)}}(\boldsymbol{k},\boldsymbol{k}')$ that we are looking for is therefore

$$\begin{aligned}
\overline{\mathcal{G}^{(3)}}(\boldsymbol{k},\boldsymbol{k}')|_{\text{“}j_1=j_2\neq j_3\text{”}} &= \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2}\mathcal{G}^{(0)}(\boldsymbol{k})U(\boldsymbol{k}-\boldsymbol{k}_1)\mathcal{G}^{(0)}(\boldsymbol{k}_1)U(\boldsymbol{k}_1-\boldsymbol{k}_2)\mathcal{G}^{(0)}(\boldsymbol{k}_2)U(\boldsymbol{k}_2-\boldsymbol{k}')\mathcal{G}^{(0)}(\boldsymbol{k}') \\
&\quad \overline{\rho(\boldsymbol{k}-\boldsymbol{k}_{)}\rho(\boldsymbol{k}_1-\boldsymbol{k}_2)\rho(\boldsymbol{k}_2-\boldsymbol{k}')}|_{\text{“}j_1=j_2\neq j_3\text{”}} \\
&= \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2}\mathcal{G}^{(0)}(\boldsymbol{k})U(\boldsymbol{k}-\boldsymbol{k}_1)\mathcal{G}^{(0)}(\boldsymbol{k}_1)U(\boldsymbol{k}_1-\boldsymbol{k}_2)\mathcal{G}^{(0)}(\boldsymbol{k}_2)U(\boldsymbol{k}_2-\boldsymbol{k}')\mathcal{G}^{(0)}(\boldsymbol{k}') \\
&\quad N^2\delta_{\boldsymbol{k},\boldsymbol{k}_2}\delta_{\boldsymbol{k},\boldsymbol{k}'} \\
&= N^2\sum_{\boldsymbol{k}_1}\mathcal{G}^{(0)}(\boldsymbol{k})U(\boldsymbol{k}-\boldsymbol{k}_1)\mathcal{G}^{(0)}(\boldsymbol{k}_1)U(\boldsymbol{k}_1-\boldsymbol{k})\mathcal{G}^{(0)}(\boldsymbol{k})U(0)\mathcal{G}^{(0)}(\boldsymbol{k})\delta_{\boldsymbol{k},\boldsymbol{k}'} \\
&\equiv \overline{\mathcal{G}^{(3)}}(\boldsymbol{k})|_{\text{“}j_1=j_2\neq j_3\text{”}}\delta_{\boldsymbol{k},\boldsymbol{k}'}.
\end{aligned} \tag{18}$$

The Feynman diagram corresponding to $\overline{\mathcal{G}^{(3)}}(\boldsymbol{k})|_{\text{“}j_1=j_2\neq j_3\text{”}}$ is shown in Fig. 1.

(b) For $n=4$ we must consider

$$\begin{aligned}
\overline{\rho(\boldsymbol{k}-\boldsymbol{k}_1)\rho(\boldsymbol{k}_1-\boldsymbol{k}_2)\rho(\boldsymbol{k}_2-\boldsymbol{k}_3)\rho(\boldsymbol{k}_3-\boldsymbol{k}')} &= \sum_{j_1}\sum_{j_2}\sum_{j_3}\sum_{j_4}\left[\prod_i\left(\frac{1}{\Omega}\int d^3R_i\right)\right] \\
& e^{-i(\boldsymbol{k}-\boldsymbol{k}_1)\cdot\boldsymbol{R}_{j_1}}e^{-i(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{R}_{j_2}}e^{-i(\boldsymbol{k}_2-\boldsymbol{k}_3)\cdot\boldsymbol{R}_{j_3}}e^{-i(\boldsymbol{k}_3-\boldsymbol{k}')\cdot\boldsymbol{R}_{j_4}}.
\end{aligned} \tag{19}$$

We want the contribution corresponding to $j_1=j_3\neq j_2=j_4$. Then the product of exponentials simplifies to

$$e^{-i(\boldsymbol{k}-\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3)\cdot\boldsymbol{R}_{j_1}}e^{-i(\boldsymbol{k}_1-\boldsymbol{k}_2+\boldsymbol{k}_3-\boldsymbol{k}')\cdot\boldsymbol{R}_{j_2}}. \tag{20}$$

Thus the average over R_{j_1} gives $\delta_{\boldsymbol{k}-\boldsymbol{k}_1,\boldsymbol{k}_3-\boldsymbol{k}_2}$, while the average over R_{j_2} gives $\delta_{\boldsymbol{k}_1-\boldsymbol{k}_2,\boldsymbol{k}'-\boldsymbol{k}_3}$. The other $N-2$ averages just give 1. Thus the total average is $\delta_{\boldsymbol{k}-\boldsymbol{k}_1,\boldsymbol{k}_3-\boldsymbol{k}_2}\delta_{\boldsymbol{k}_1-\boldsymbol{k}_2,\boldsymbol{k}'-\boldsymbol{k}_3}$ which can be rewritten $\delta_{\boldsymbol{k}-\boldsymbol{k}_1,\boldsymbol{k}_3-\boldsymbol{k}_2}\delta_{\boldsymbol{k},\boldsymbol{k}'}$. The site summations for this contribution give

$$\sum_{j_1}\sum_{j_2}\sum_{j_3}\sum_{j_4}\delta_{j_1,j_3}\delta_{j_2,j_4}(1-\delta_{j_1,j_2}) = N^2 - N \approx N^2. \tag{21}$$

Thus the contribution to the impurity average is $N^2\delta_{\boldsymbol{k}-\boldsymbol{k}_1,\boldsymbol{k}_3-\boldsymbol{k}_2}\delta_{\boldsymbol{k},\boldsymbol{k}'}$. Considering the impurity average of Eq. (120) in the notes, the contribution to $\overline{\mathcal{G}^{(4)}}(\boldsymbol{k},\boldsymbol{k}')$ that we are looking for is therefore

$$\begin{aligned}
\overline{\mathcal{G}^{(4)}}(\boldsymbol{k},\boldsymbol{k}')|_{\text{``}j_1=j_3\neq j_2=j_4''} &= \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3}\mathcal{G}^{(0)}(\boldsymbol{k})U(\boldsymbol{k}-\boldsymbol{k}_1)\mathcal{G}^{(0)}(\boldsymbol{k}_1)U(\boldsymbol{k}_1-\boldsymbol{k}_2)\mathcal{G}^{(0)}(\boldsymbol{k}_2) \\
&\times\ U(\boldsymbol{k}_2-\boldsymbol{k}_3)\mathcal{G}^{(0)}(\boldsymbol{k}_3)U(\boldsymbol{k}_3-\boldsymbol{k}')\mathcal{G}^{(0)}(\boldsymbol{k}') \\
&\times\ \overline{\rho(\boldsymbol{k}-\boldsymbol{k}_1)\rho(\boldsymbol{k}_1-\boldsymbol{k}_2)\rho(\boldsymbol{k}_2-\boldsymbol{k}_3)\rho(\boldsymbol{k}_3-\boldsymbol{k}')}|_{\text{``}j_1=j_3\neq j_2=j_4\text{''}} \\
&= \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3}\mathcal{G}^{(0)}(\boldsymbol{k})U(\boldsymbol{k}-\boldsymbol{k}_1)\mathcal{G}^{(0)}(\boldsymbol{k}_1)U(\boldsymbol{k}_1-\boldsymbol{k}_2)\mathcal{G}^{(0)}(\boldsymbol{k}_2) \\
&\times\ U(\boldsymbol{k}_2-\boldsymbol{k}_3)\mathcal{G}^{(0)}(\boldsymbol{k}_3)U(\boldsymbol{k}_3-\boldsymbol{k}')\mathcal{G}^{(0)}(\boldsymbol{k}') \\
&\times\ N^2\delta_{\boldsymbol{k}-\boldsymbol{k}_1,\boldsymbol{k}_3-\boldsymbol{k}_2}\delta_{\boldsymbol{k},\boldsymbol{k}'} \\
&= N^2\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2}\mathcal{G}^{(0)}(\boldsymbol{k})U(\boldsymbol{k}-\boldsymbol{k}_1)\mathcal{G}^{(0)}(\boldsymbol{k}_1)U(\boldsymbol{k}_1-\boldsymbol{k}_2)\mathcal{G}^{(0)}(\boldsymbol{k}_2) \\
&\times\ U(\boldsymbol{k}_1-\boldsymbol{k})\mathcal{G}^{(0)}(\boldsymbol{k}-\boldsymbol{k}_1+\boldsymbol{k}_2)U(\boldsymbol{k}_2-\boldsymbol{k}_1)\mathcal{G}^{(0)}(\boldsymbol{k})\delta_{\boldsymbol{k},\boldsymbol{k}'} \\
&\equiv \overline{\mathcal{G}^{(4)}}(\boldsymbol{k})|_{\text{``}j_1=j_3\neq j_2=j_4\text{''}}\delta_{\boldsymbol{k},\boldsymbol{k}'}.
\end{aligned} \tag{22}$$

The Feynman diagram corresponding to $\overline{\mathcal{G}^{(4)}}(\boldsymbol{k})|_{\text{``}j_1=j_3\neq j_2=j_4\text{''}}$ is shown in Fig. 2. While the labeling of wavevectors in this diagram is perfectly correct and fine, it is also possible to use an alternative and equivalent labeling of wavevectors that is somewhat more "symmetrical". This alternative labeling is shown in Fig. 3 and was obtained by replacing the summation variable $\boldsymbol{k}_2$ by a different summation variable $\boldsymbol{k}_2' \equiv \boldsymbol{k}+\boldsymbol{k}_2-\boldsymbol{k}_1$ and then renaming $\boldsymbol{k}_2'$ as $\boldsymbol{k}_2$. This corresponds to the following alternative and equivalent expression for $\overline{\mathcal{G}^{(4)}}(\boldsymbol{k},\boldsymbol{k}')|_{\text{``}j_1=j_3\neq j_2=j_4\text{''}}$:

$$\begin{aligned}
\overline{\mathcal{G}^{(4)}}(\boldsymbol{k},\boldsymbol{k}')|_{\text{``}j_1=j_3\neq j_2=j_4\text{''}} &= N^2\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2}\mathcal{G}^{(0)}(\boldsymbol{k})U(\boldsymbol{k}-\boldsymbol{k}_1)\mathcal{G}^{(0)}(\boldsymbol{k}_1)U(\boldsymbol{k}-\boldsymbol{k}_2)\mathcal{G}^{(0)}(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}) \\
&\times\ U(\boldsymbol{k}_1-\boldsymbol{k})\mathcal{G}^{(0)}(\boldsymbol{k}_2)U(\boldsymbol{k}_2-\boldsymbol{k})\mathcal{G}^{(0)}(\boldsymbol{k})\delta_{\boldsymbol{k},\boldsymbol{k}'}.
\end{aligned} \tag{23}$$

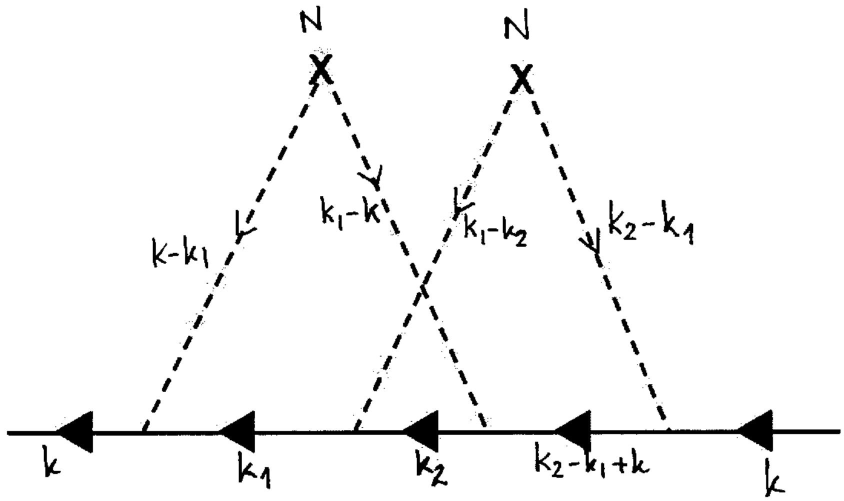


Figure 2: The Feynman diagram for problem 2(b).

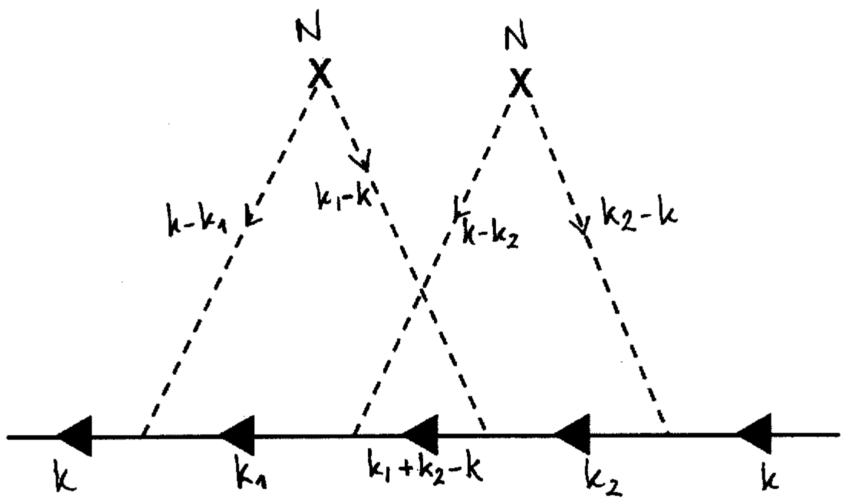


Figure 3: The Feynman diagram for problem 2(b) with an alternative labeling of wavevectors.

3. Impurity scattering: Feynman diagrams at order $n = 4$.

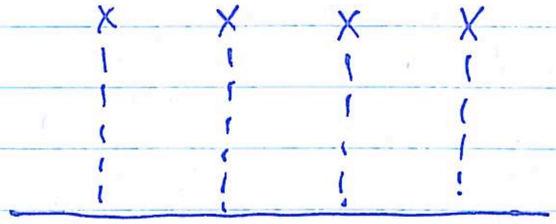
The Feynman diagrams are given below. To understand where they come from it may be helpful to keep in mind the impurity average at order $n = 4$, which reads

$$\begin{aligned}&\overline{\rho(\boldsymbol{k}-\boldsymbol{k}_1)\rho(\boldsymbol{k}_1-\boldsymbol{k}_2)\rho(\boldsymbol{k}_2-\boldsymbol{k}_3)\rho(\boldsymbol{k}_3-\boldsymbol{k}')}\\ &=\sum_{j_1}\sum_{j_2}\sum_{j_3}\sum_{j_4}\prod_i\left(\frac{1}{\Omega}\int d^3R_i\right)e^{-i(\boldsymbol{k}-\boldsymbol{k}_1)\cdot\boldsymbol{R}_{j_1}}e^{-i(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{R}_{j_2}}e^{-i(\boldsymbol{k}_2-\boldsymbol{k}_3)\cdot\boldsymbol{R}_{j_3}}e^{-i(\boldsymbol{k}_3-\boldsymbol{k}')\cdot\boldsymbol{R}_{j_4}}.\end{aligned} \tag{24}$$

The various Feynman diagrams are associated with the various ways in which the impurities labeled j_1, j_2, j_3 and j_4 are different or the same. I have indicated this characteristic next to each of the diagrams below. The number of impurity crosses, called m, is also indicated. There turns out to be 15 different Feynman diagrams at this order.

All impurities different:

$j_1 \neq j_2 \neq j_3 \neq j_4$:

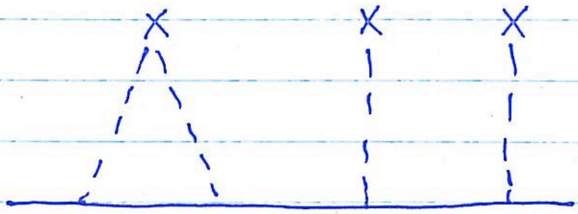


1

$(m=4)$

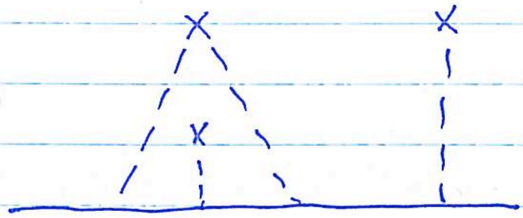
Two the same, the others different:

$j_1 = j_2 \neq j_3 \neq j_4$:



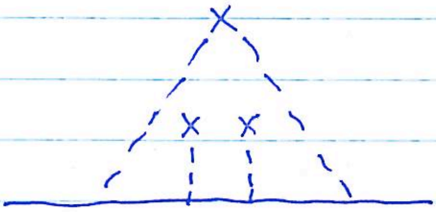
2

$j_1 = j_3 \neq j_2 \neq j_4$:



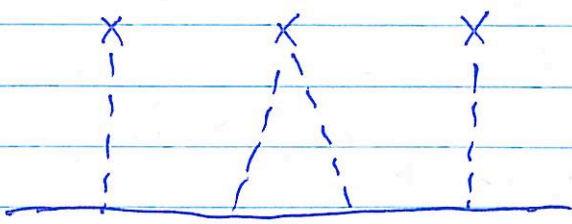
3

$j_1 = j_4 \neq j_2 \neq j_3$:



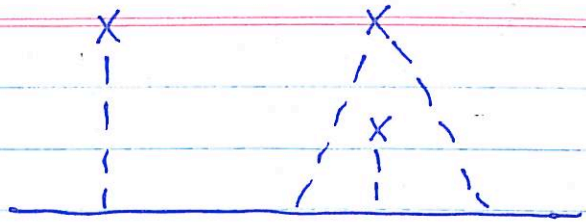
4

$j_2 = j_3 \neq j_1 \neq j_4$:



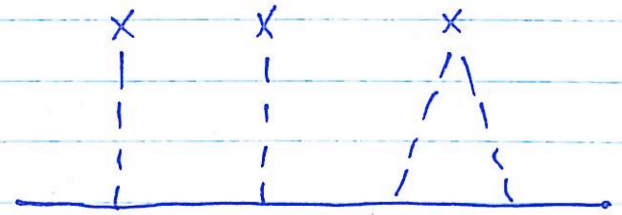
5

$j_2 = j_4 \neq j_1 \neq j_3$:



6

$j_3 = j_4 \neq j_1 \neq j_2$:

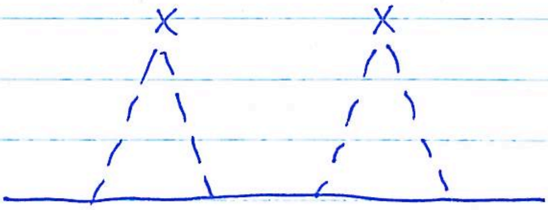


7

(diagrams 2 - 7 have $m = 3$)

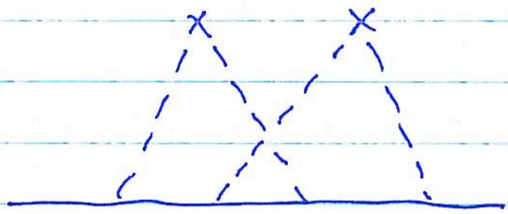
<u>Two the same + two the same :</u>

$j_1 = j_2 \neq j_3 = j_4$:



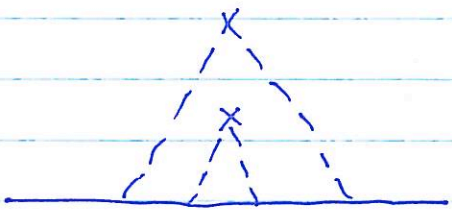
8

$j_1 = j_3 \neq j_2 = j_4$:



9

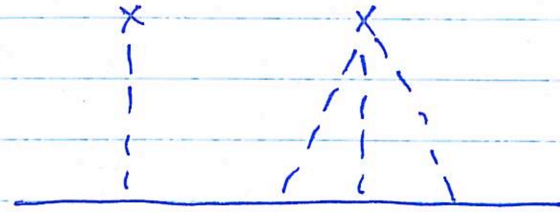
$j_1 = j_4 \neq j_2 = j_3$:

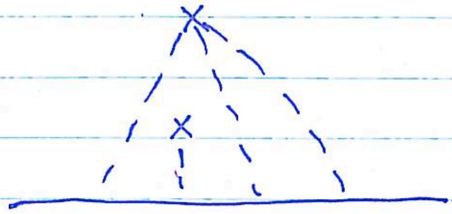


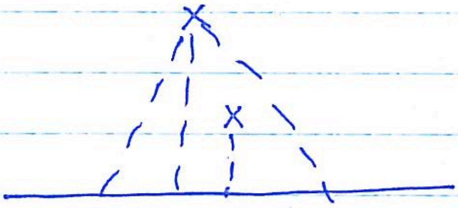
10

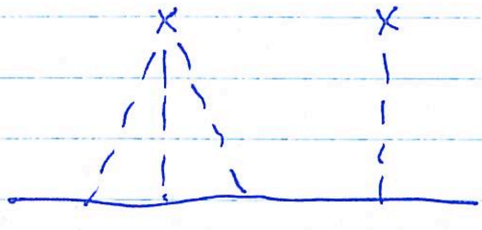
(diagrams 8 - 10 have $m = 2$)

Three the same, one different:

j_1 different:  (11)

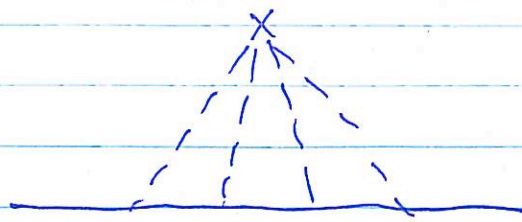
j_2 different:  (12)

j_3 different:  (13)

j_4 different:  (14)

(diagrams (11) - (14) have $m=2$)

All four the same:

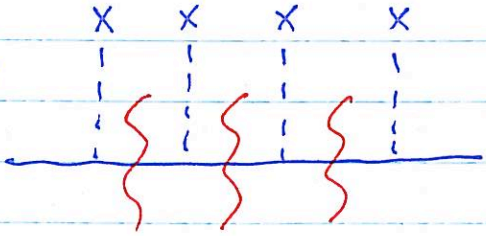
$j_1 = j_2 = j_3 = j_4$:  (15)

$(m=1)$

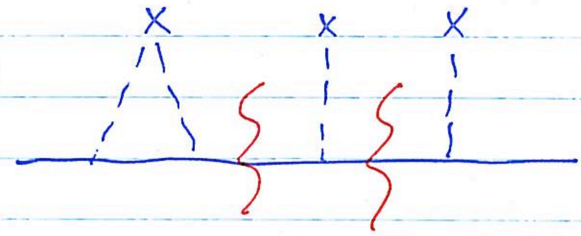
The irreducible diagrams are (4), (9), (10), (12), (13), and (15). Thus there are 6 irreducible diagrams at order $n=4$.

The remaining 9 diagrams are reducible because they can be cut into two separate pieces by cutting only a single internal electron line. (Some of these diagrams can be cut like this in more than one way. We indicate which lines can be cut with a red wavy line in the following figures).

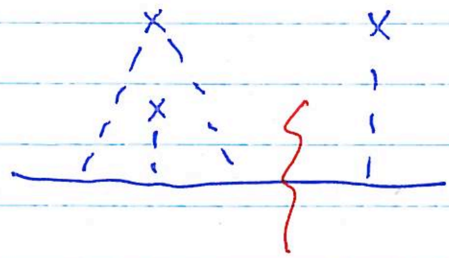
(1):



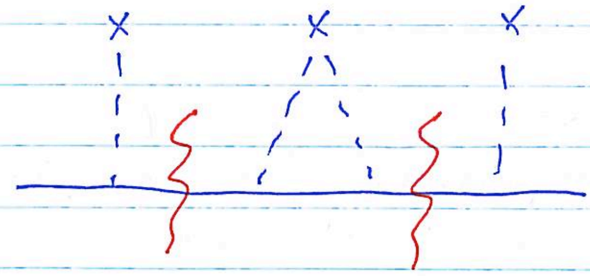
(2):



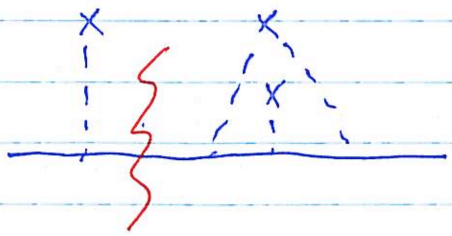
(3):



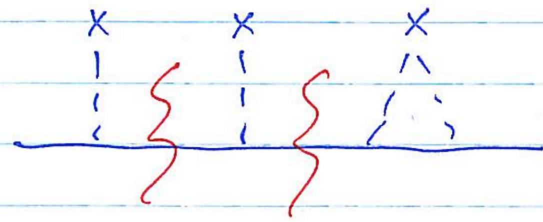
(5):



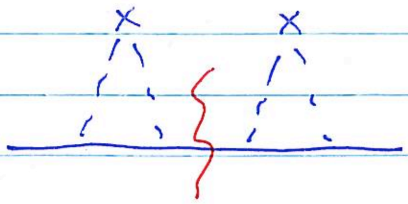
(6):



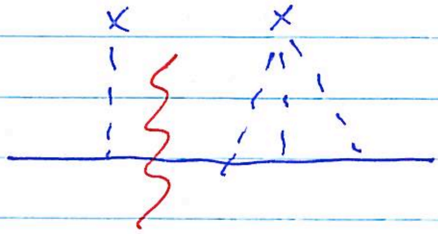
(7):



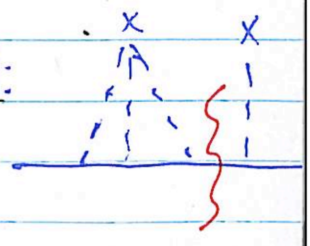
(8):



(11):

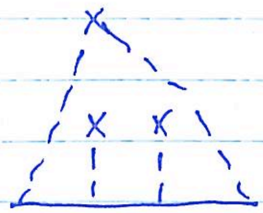


(14):

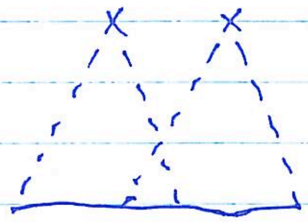


Since there are 6 irreducible diagrams, there are 6 self-energy diagrams which we name by putting a prime (′) after the number for the irreducible diagram they arise from:

4′ :



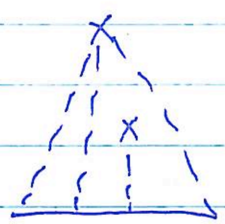
9′ :



10′ :

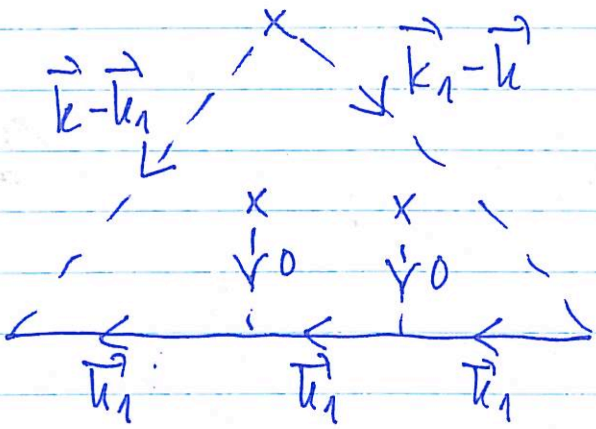
12′ :

13′ :

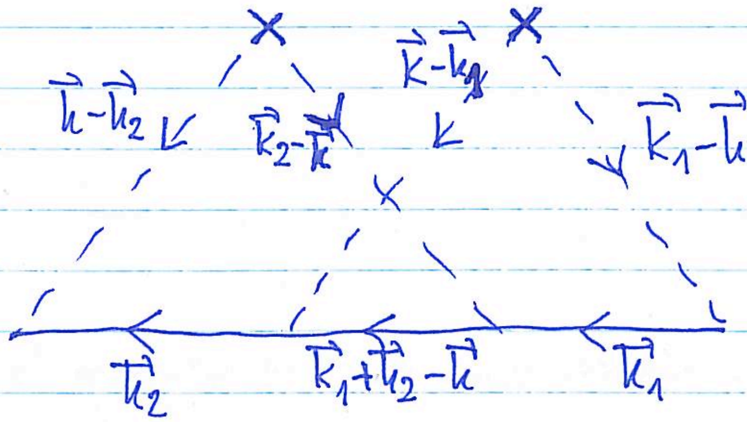


15′ :

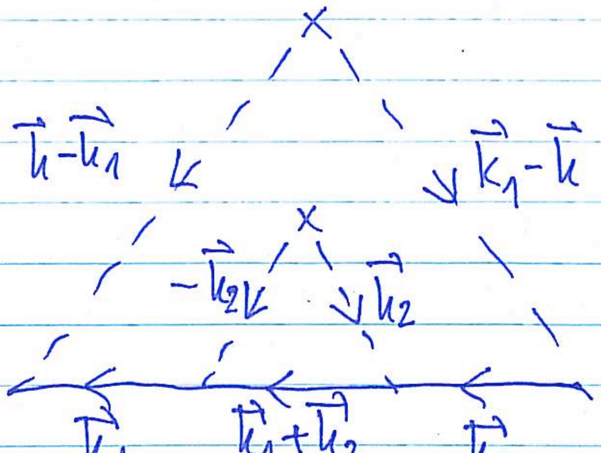
The mathematical expressions for these self-energy diagrams can be written down after putting on momentum labels on the electron lines and interaction lines according to the Feynman rules:



(4')

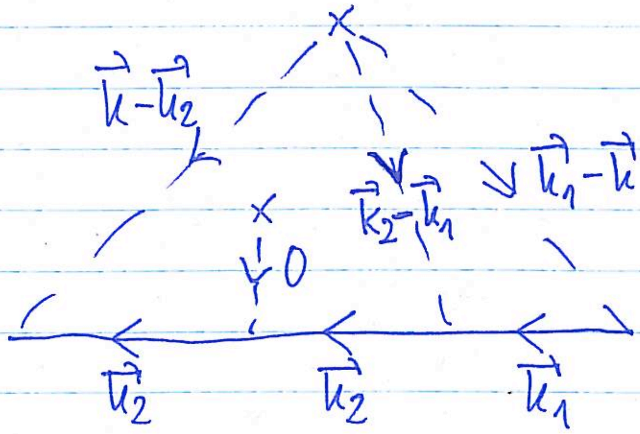


(9')

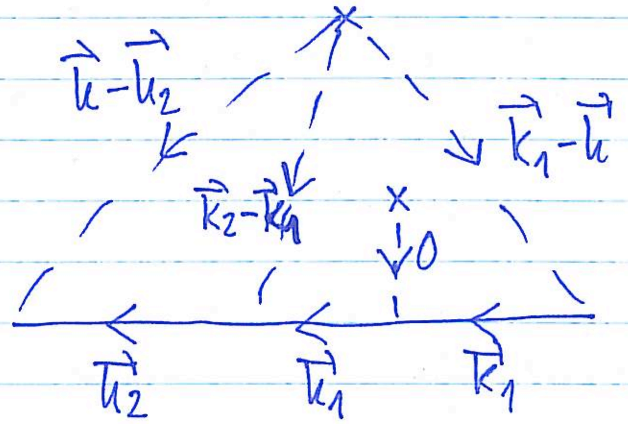


(10')

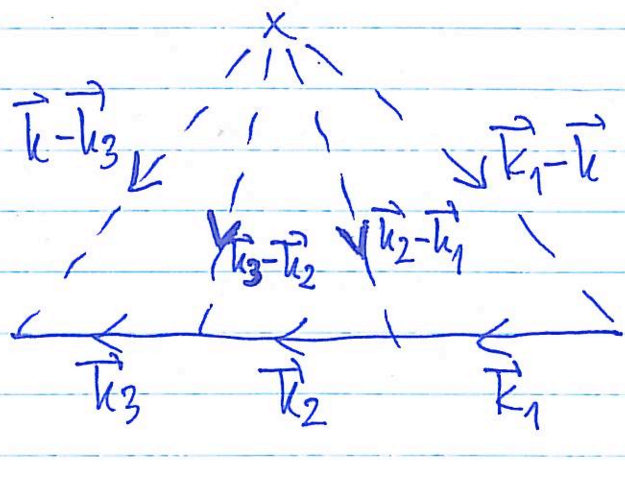
7


$\vec{k}-\vec{k}_2$
$\vec{k}_2-\vec{k}_1$
$\vec{k}_1-\vec{k}$
0
$\vec{k}_2$
$\vec{k}_2$
$\vec{k}_1$

12'


$\vec{k}-\vec{k}_2$
$\vec{k}_1-\vec{k}$
$\vec{k}_2-\vec{k}_1$
0
$\vec{k}_2$
$\vec{k}_1$
$\vec{k}_1$

13'


$\vec{k}-\vec{k}_3$
$\vec{k}_1-\vec{k}$
$\vec{k}_3-\vec{k}_2$
$\vec{k}_2-\vec{k}_1$
$\vec{k}_3$
$\vec{k}_2$
$\vec{k}_1$

15'

Note that in these figures the momentum is conserved at every point where an interaction line meets the electron lines, and also at every impurity cross (i.e. the sum of momenta going out from each impurity cross is zero). Based on these figures, we find the following expressions for the 6 self-energy diagrams:

$$\begin{aligned}
\Sigma^{(4')}(\boldsymbol{k}) &= N^3(U(0))^2 \sum_{\boldsymbol{k}_1} U(\boldsymbol{k}-\boldsymbol{k}_1)U(\boldsymbol{k}_1-\boldsymbol{k})(\mathcal{G}^{(0)}(\boldsymbol{k}_1))^3, \\
\Sigma^{(9')}(\boldsymbol{k}) &= N^2 \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2} U(\boldsymbol{k}-\boldsymbol{k}_2)U(\boldsymbol{k}_2-\boldsymbol{k})U(\boldsymbol{k}-\boldsymbol{k}_1)U(\boldsymbol{k}_1-\boldsymbol{k})\mathcal{G}^{(0)}(\boldsymbol{k}_1)\mathcal{G}^{(0)}(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k})\mathcal{G}^{(0)}(\boldsymbol{k}_2), \\
\Sigma^{(10')}(\boldsymbol{k}) &= N^2 \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2} U(\boldsymbol{k}-\boldsymbol{k}_1)U(\boldsymbol{k}_1-\boldsymbol{k})U(-\boldsymbol{k}_2)U(\boldsymbol{k}_2)(\mathcal{G}^{(0)}(\boldsymbol{k}_1))^2\mathcal{G}^{(0)}(\boldsymbol{k}_1+\boldsymbol{k}_2), \\
\Sigma^{(12')}(\boldsymbol{k}) &= N^2U(0) \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2} U(\boldsymbol{k}-\boldsymbol{k}_2)U(\boldsymbol{k}_2-\boldsymbol{k}_1)U(\boldsymbol{k}_1-\boldsymbol{k})(\mathcal{G}^{(0)}(\boldsymbol{k}_2))^2\mathcal{G}^{(0)}(\boldsymbol{k}_1), \\
\Sigma^{(13')}(\boldsymbol{k}) &= N^2U(0) \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2} U(\boldsymbol{k}-\boldsymbol{k}_2)U(\boldsymbol{k}_2-\boldsymbol{k}_1)U(\boldsymbol{k}_1-\boldsymbol{k})\mathcal{G}^{(0)}(\boldsymbol{k}_2)(\mathcal{G}^{(0)}(\boldsymbol{k}_1))^2, \\
\Sigma^{(15')}(\boldsymbol{k}) &= N \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3} U(\boldsymbol{k}-\boldsymbol{k}_3)U(\boldsymbol{k}_3-\boldsymbol{k}_2)U(\boldsymbol{k}_2-\boldsymbol{k}_1)U(\boldsymbol{k}_1-\boldsymbol{k})\mathcal{G}^{(0)}(\boldsymbol{k}_1)\mathcal{G}^{(0)}(\boldsymbol{k}_2)\mathcal{G}^{(0)}(\boldsymbol{k}_3).
\end{aligned}$$

Note that in each expression, the number of internal wavevector summations is given by $n-m$ where n is the order of the diagram (i.e. $n = 4$ here) and m is the number of impurity crosses in the diagram.

Other correct ways of labeling the momenta (i.e., wavevectors) than those shown on the previous pages are also possible. The resulting expressions will then superficially look different from those above, but can be shown to be identical by renaming the dummy summation variables (the wavevectors being summed over).