TFY4210, Quantum theory of many-particle systems, 2015: Tutorial 13

1. Dirac equation in two spatial dimensions with an external magnetic field.

In this problem we consider the Dirac equation in two spatial dimensions (described by cartesian coordinates x and y). We set $c = 1$ and $\hbar = 1$ for simplicity.

(a) Show that a valid representation (to be used in the following) for the α and β matrices is

$$\beta = \sigma_3, \quad \alpha_1 = \sigma_1, \quad \alpha_2 = \sigma_2. \tag{1}$$

(b) An external magnetic field of magnitude B is applied in the z direction (i.e. perpendicular to the plane of motion): $\boldsymbol{B} = B\hat{z}$. In terms of electromagnetic potentials this can be represented as $\phi = 0$, $\boldsymbol{A} = (0, Bx, 0)$. Show that the Dirac equation for a particle of charge q in the presence of the magnetic field can be rewritten as

$$(i\sigma_3\partial_t - \sigma_2\partial_x + \sigma_1\partial_y - iqBx\sigma_1 - m)\Psi = 0. \tag{2}$$

(c) Explain why it is natural to use the ansatz (here p_y is a number)

$$\Psi(x, y, t) \quad = \quad e^{ip_y y - iEt}\begin{pmatrix} f(x) \\ g(x) \end{pmatrix}. \tag{3}$$

(d) Use the ansatz to derive an equation of the form $Q\begin{pmatrix} f(x) \\ g(x) \end{pmatrix} = 0$, where Q is a 2×2 matrix. Simplify expressions by introducing the operators

$$\xi_\pm = -i\partial_x \mp i(p_y - qBx). \tag{4}$$

(e) Eliminate the function $g(x)$ from the problem to obtain an equation for $f(x)$ alone.

(f) Solve this equation, thus finding the solutions for both $f(x)$ and E (hint: harmonic oscillator). Finally, find $g(x)$.

2. Green functions for a Heisenberg ferromagnet.

In this problem we consider a Heisenberg ferromagnet. We write the Hamiltonian as

$$H = \frac{1}{2}\sum_{i,j} J_{ij}\boldsymbol{S}_i \cdot \boldsymbol{S}_j = \frac{1}{2}\sum_{i,j} J_{ij}\left[\frac{1}{2}(S_i^+S_j^- + S_i^-S_j^+) + S_i^zS_j^z\right] \tag{5}$$

with $J_{ij} \leq 0$. We also assume that J_{ij} is only a function of $\boldsymbol{r}_i - \boldsymbol{r}_j$, so that H is translationally invariant. Let us define the retarded Green function

$$R_{ij}(t) \equiv -i\theta(t)\langle[S_i^-(t), S_j^+(0)]\rangle \tag{6}$$

where the time evolution of the operators is defined as $A(t) \equiv e^{iHt}A(0)e^{-iHt}$.

(a) Show that $R_{ij}(t)$ satisfies the equation of motion

$$i\frac{\partial R_{ij}(t)}{\partial t} = -2\delta_{ij}\delta(t)\langle S_i^z\rangle - i\theta(t)\langle[[S_i^-(t), H], S_j^+(0)]\rangle. \tag{7}$$

(b) Show that

$$[S_i^-(t), H] = -\sum_{j'} J_{ij'}(S_{j'}^- S_i^z(t) - S_{j'}^z(t)S_i^-(t)). \tag{8}$$

(c) Upon inserting (8) into (7), the rhs can be seen to involve expressions of the type $-i\theta(t)\langle[S_i^z(t)S_{j'}^-(t), S_j^+(0)]\rangle$ which are retarded Green functions of higher order than $R_{ij}(t)$. To make progress, we will approximate these higher-order Green functions in terms of the R-functions. This will be done by replacing the operator S_j^z by its expectation value $\langle S^z\rangle$ (here we omitted the site index in the expectation value, since this becomes site-independent due to the translational invariance of the Hamiltonian). Show that this "mean-field" approximation gives

$$i\frac{\partial R_{ij}(t)}{\partial t} = -\langle S^z\rangle \left[2\delta_{ij}\delta(t) + \sum_{j'} J_{ij'}(R_{j'j}(t) - R_{ij}(t))\right]. \tag{9}$$

(d) Next, we specialize to zero temperature, for which the ferromagnetic ground state has $\langle S^z\rangle = S$, where S is the total quantum number of the spins. Introduce $R(\boldsymbol{k}, t)$ as the space Fourier transform of $R_{ij}(t)$ and show that it satisfies

$$i\frac{\partial R(\boldsymbol{k}, t)}{\partial t} = -2S\delta(t) - \Omega_{\boldsymbol{k}} R(\boldsymbol{k}, t) \tag{10}$$

where we defined

$$\Omega_{\boldsymbol{k}} \equiv S(J_{\boldsymbol{k}} - J_0) \tag{11}$$

where $J_{\boldsymbol{k}}$ is the Fourier transform of J_{ij}.

(e) Introduce $R(\boldsymbol{k}, \omega)$ as the time Fourier transform of $R(\boldsymbol{k}, t)$ and show that it is given by

$$R(\boldsymbol{k}, \omega) = \frac{-2S}{\omega + \Omega_{\boldsymbol{k}} + i\eta} \tag{12}$$

where $\eta = 0^+$ (introduced to ensure that $R(\boldsymbol{k}, t) \propto \theta(t)$).

(f) Specialize now to the case of nearest-neighbour interactions on a hypercubic lattice, i.e. $J_{ij} = -|J|$ for all nearest neighbours and 0 otherwise. Show that $\Omega_{\boldsymbol{k}} = 2|J|S\sum_{\boldsymbol{\delta}}(1 - \cos \boldsymbol{k}\cdot\boldsymbol{\delta})$ which is nothing but the spin-wave dispersion $\omega_{\boldsymbol{k}}$ that we found earlier in the course when studying this problem using spin-wave theory.