NTNU – Trondheim
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Department of Physics

Examination paper for
TFY4240 Electromagnetic theory

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Permitted examination support material: C

Approved calculator

Rottmann: Matematisk Formelsamling (or an equivalent book of mathematical formulas)

Other information:

This exam consists of three problems, each containing several subproblems. In many cases it is possible to solve later subproblems even if earlier subproblems were not solved. Some formulas can be found on the pages following the problems.

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Date Signature

Problem 1.

A sphere of radius R and with zero net charge is placed in a uniform external electric field of magnitude E_0. The entire sphere, including its interior, is made of perfectly conducting material, and outside the sphere is vacuum. We choose a coordinate system (see Fig. 1) with the origin at the center of the sphere and the z axis pointing in the direction of the external electric field, which thus can be written $\boldsymbol{E}_0 = E_0\hat{\boldsymbol{z}}$.

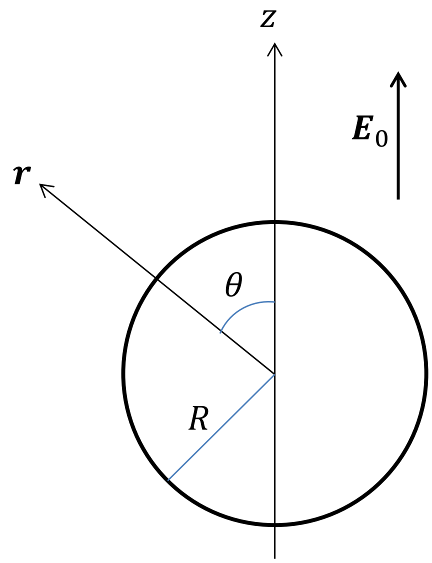

Figure 1

a) What is the electric field inside the conducting sphere?

b) What are the boundary conditions for the potential $V(\boldsymbol{r})$ (i) on the surface of the sphere ($|\boldsymbol{r}| = R$), (ii) as $|\boldsymbol{r}| \to \infty$?

In the following we wish to find the potential $V(\boldsymbol{r})$ at an arbitrary point $\boldsymbol{r}$ (with spherical coordinates (r, θ, φ)) outside the sphere. Due to the symmetry of the problem, $V(\boldsymbol{r})$ will be independent of φ and can be expanded as

$$V(\boldsymbol{r}) = \sum_{\ell=0} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta) \tag{1}$$

where $P_\ell(x)$ is the Legendre polynomial of degree ℓ in the variable $x = \cos\theta$ (in particular, $P_0(x) = 1$, $P_1(x) = x$).

c) Show that the potential outside the sphere can be expressed as

$$V(\boldsymbol{r}) = A_0 + \frac{B_0}{r} + C(r)\cos\theta, \tag{2}$$

and give the expression for the function $C(r)$.

d) Find an expression for the surface charge density σ on the sphere. Use the fact that the sphere has zero net charge to show that $B_0 = 0$.

e) Calculate the induced electric dipole moment $\boldsymbol{p} = \int d^3r\, \boldsymbol{r}\rho(\boldsymbol{r})$ of the sphere. [Hint: By symmetry, $\boldsymbol{p}$ must point along the z axis, so it is sufficient to calculate p_z. Use the fact that the induced charge is located at the surface to first rewrite the integral as a surface integral involving the surface charge density σ.]

Problem 2.

Consider Maxwell's equations in a linear non-conducting medium with permittivity ϵ and permeability μ, and no free charges or currents ($\rho_\mathrm{f} = \boldsymbol{J}_\mathrm{f} = 0$).

a) Show that $\boldsymbol{E}$ and $\boldsymbol{B}$ each satisfy the wave equation, and find an expression for the wave velocity v.

b) Consider the plane wave solution

$$\tilde{\boldsymbol{E}} = \tilde{\boldsymbol{E}}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}, \quad \tilde{\boldsymbol{B}} = \tilde{\boldsymbol{B}}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}, \tag{3}$$

of the wave equations for $\boldsymbol{E}$ and $\boldsymbol{B}$, where $\omega = vk$. Show the following properties of these plane waves:

- $\boldsymbol{E}$ and $\boldsymbol{B}$ are in phase
- the directions of $\boldsymbol{E}_0$, $\boldsymbol{B}_0$, and $\boldsymbol{k}$ form a right-handed coordinate system
- $|\boldsymbol{B}| = |\boldsymbol{E}|/v$

How does the intensity of the electromagnetic wave depend on E_0? Explain your reasoning.

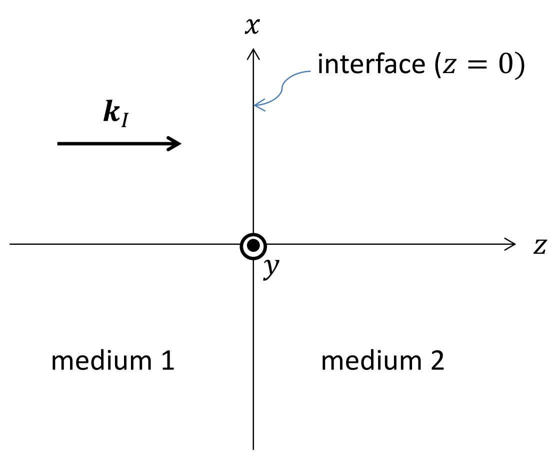

Figure 2

Next, consider two linear non-conducting media 1 and 2 with wave velocities v_1 and v_2, respectively. The two media are separated by a flat interface. A plane electromagnetic wave in medium 1 is incident on the interface, giving rise to a reflected wave in medium 1 and a transmitted wave in medium 2. The wavevector of the incident wave is perpendicular to the interface (i.e., we are considering "normal incidence").

We choose a coordinate system such that the interface is the xy plane (i.e., at $z = 0$), with medium 1 in the region $z < 0$ and medium 2 in the region $z > 0$. The incident wave has (angular) frequency ω and wavevector $\boldsymbol{k}_I = k_1\hat{\boldsymbol{z}}$ where $k_1 > 0$ (see Fig. 2). Taking the incident wave to be polarized along the x direction, the electric and magnetic fields of the incident wave can be written

$$\tilde{\boldsymbol{E}}_I(z,t) = \tilde{E}_{0I}\, e^{i(k_1 z-\omega t)}\,\hat{\boldsymbol{x}} \quad \text{and} \quad \tilde{\boldsymbol{B}}_I(z,t) = \frac{1}{v_1}\tilde{E}_{0I}\, e^{i(k_1 z-\omega t)}\,\hat{\boldsymbol{y}}. \tag{4}$$

c) Write down the expressions for $\boldsymbol{E}$ and $\boldsymbol{B}$ analogous to Eq. (4) for (i) the reflected wave (ii) the transmitted wave. [You may assume without proof that also these waves are polarized along the x direction.]

d) Find an expression for the reflection coefficient R (the ratio of the reflected and incident intensities) in terms of the permittivities and permeabilities of the two media (i.e., ϵ_1, μ_1, ϵ_2, μ_2).

Problem 3.

In the Lorenz gauge, the scalar and vector potentials can be expressed as

$$V(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3r'\,\frac{\rho(\boldsymbol{r}',t_r)}{|\boldsymbol{r}-\boldsymbol{r}'|}, \quad \boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\int d^3r'\,\frac{\boldsymbol{J}(\boldsymbol{r}',t_r)}{|\boldsymbol{r}-\boldsymbol{r}'|}. \tag{5}$$

One calls these the retarded potentials, because the time t_r, known as the **retarded time**, appears in the integrands.

a) Give the mathematical expression for the retarded time t_r, and use this to give a physical interpretation of how the charges and currents contribute to the potentials, as expressed by Eq. (5).

In the following we consider a straight, infinitely long wire. At time $t = 0$ a constant current I_0 is abruptly turned on in the wire, so that the wire current $I(t)$ can be written

$$I(t) = \begin{cases} 0 & \text{for } t \le 0, \\ I_0 & \text{for } t > 0. \end{cases} \tag{6}$$

We wish to find the resulting electric and magnetic fields at an arbitrary time $t > 0$ at an arbitrary point $\boldsymbol{r}$ outside the wire.

Because of the symmetry of the problem, we choose a cylindrical coordinate system in which $\boldsymbol{r}$ has coordinates (s, φ, z), and the z axis is chosen to coincide with the wire, with the z direction being the direction of the current (see Fig. 3).

b) What is the scalar potential $V(\boldsymbol{r},t)$ in this problem? Explain your reasoning.

c) Show that the vector potential becomes $\boldsymbol{A}(\boldsymbol{r},t) = A_z(\boldsymbol{r},t)\,\hat{\boldsymbol{z}}$, with

$$A_z(\boldsymbol{r},t) = \begin{cases} 0 & \text{for } t < s/c, \\ \frac{\mu_0 I_0}{2\pi}\ln\left(\frac{ct+\sqrt{(ct)^2-s^2}}{s}\right) & \text{for } t > s/c. \end{cases} \tag{7}$$

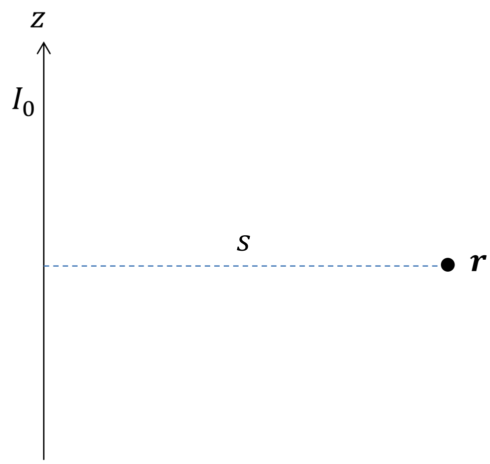

Figure 3

[Hint: Because of the symmetry of the problem, there can be no dependence on the z coordinate of $\boldsymbol{r}$, so you may set $z = 0$ for convenience. For $t > s/c$, first argue that only a finite segment of the wire will contribute to the integral, and use this to determine the integration limits for z'.]

d) Find the electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ and the magnetic field $\boldsymbol{B}(\boldsymbol{r}, t)$. (Using the cylindrical coordinate system is again recommended.)

e) Consider the expressions for $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$ in the limit $t \to \infty$. Are the results reasonable? Explain.

Formulas

Some formulas that you may or may not need (you should know the meaning of the symbols and possible limitations of validity):

$$\int_{-1}^{1} dx\, P_\ell(x) P_{\ell'}(x) = \frac{2}{2\ell+1}\delta_{\ell,\ell'} \tag{8}$$

$$\sigma = -\epsilon_0 \left[\frac{\partial V}{\partial n}\bigg|_{\text{outside}} - \frac{\partial V}{\partial n}\bigg|_{\text{inside}} \right] \tag{9}$$

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \tag{10}$$

$$\boldsymbol{D}_1^\perp = \boldsymbol{D}_2^\perp, \quad \boldsymbol{B}_1^\perp = \boldsymbol{B}_2^\perp, \quad \boldsymbol{E}_1^\parallel = \boldsymbol{E}_2^\parallel, \quad \boldsymbol{H}_1^\parallel = \boldsymbol{H}_2^\parallel \tag{11}$$

FUNDAMENTAL CONSTANTS

ϵ_0	$=$	$8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$	(permittivity of free space)
μ_0	$=$	$4\pi \times 10^{-7}\,\mathrm{N/A^2}$	(permeability of free space)
c	$=$	$3.00 \times 10^{8}\,\mathrm{m}/s$	(speed of light)
e	$=$	$1.60 \times 10^{-19}\,\mathrm{C}$	(charge of the electron)
m	$=$	$9.11 \times 10^{-31}\,\mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general :

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter :

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \dfrac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions :

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \dfrac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media :

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \dfrac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy : $U = \dfrac{1}{2}\displaystyle\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) d\tau$

Momentum : $\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d\tau$

Poynting vector : $\mathbf{S} = \dfrac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$

Larmor formula : $P = \dfrac{\mu_0}{6\pi c} q^2 a^2$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem : $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $d\tau = dx\,dy\,dz$

$$\textit{Gradient}: \quad \nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

$$\textit{Divergence}: \quad \nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\textit{Curl}: \quad \nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

$$\textit{Laplacian}: \quad \nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

$$\textit{Gradient}: \quad \nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

$$\textit{Divergence}: \quad \nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

$$\textit{Curl}: \quad \nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$$

$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

$$\textit{Laplacian}: \quad \nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $d\tau = s\,ds\,d\phi\,dz$

$$\textit{Gradient}: \quad \nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

$$\textit{Divergence}: \quad \nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

$$\textit{Curl}: \quad \nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

$$\textit{Laplacian}: \quad \nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$