TFY4240 Problem set 3 Autumn 2015



Problem 1.

Let f(x) be some function defined on the interval [-1, 1].

a) Show that f(x) can be expanded in terms of the Legendre polynomials $P_{\ell}(x)$ as

$$f(x) = \sum_{\ell=0}^{\infty} A_{\ell} P_{\ell}(x), \qquad (1a)$$

where

$$A_{\ell} = \frac{2\ell + 1}{2} \int_{-1}^{1} dx \, f(x) P_{\ell}(x).$$
 (1b)

Hint: Use the orthogonality of the Legendre polynomials.

b) For the particular case that

$$f(x) = \begin{cases} -1 & x < 0\\ +1 & x > 0 \end{cases}$$
(2)

argue without doing any calculation that the expansion (1a) only will contain contributions from the polynomials $P_{2n+1}(x)$ with n = 0, 1, 2, ...

c) For f(x) given by Eq. (2) show that the first few terms of the expansion read

$$f(x) = \frac{3}{2}P_1(x) - \frac{7}{8}P_3(x) + \frac{11}{16}P_5(x) + \dots$$
(3)

Problem 2.

The (electrical) potential on the surface of a sphere of radius R is given by

$$V(R,\theta) = V_0 \cos^2 \theta,$$

where V_0 is some positive constant.

a) Show that the potential outside the sphere is given by

$$V(r,\theta) = \frac{V_0}{3} \left[\frac{R}{r} + 2\left(\frac{R}{r}\right)^3 P_2(\cos\theta) \right].$$

- **b**) Determine the surface charge distribution on the sphere.
- c) Integrate over the surface charge to find the quadrupole moment of the sphere. Check to see that this agrees with the l = 2 term in the potential outside the sphere.

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Problem 3.

This problem is devoted to the study of electrostatics for various geometries, and we will apply the method of images.

a) Describe in words what is meant by the method of images and for what region the scalar potential obtained by this method can be used.



A change q > 0 is situated a distance $\Delta z = h > 0$ above a grounded conducting half space. A coordinate system is defined so that the half space is located in the region $z \leq R$, and the charge is positioned on the z-axis at height z = R + h (see figure).

In the lectures (or in the book), it was shown that the scalar potential for this system (valid for $z \ge R$) is given by

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{|\mathbf{r} - (R+h)\hat{\mathbf{z}}|} - \frac{1}{4\pi\varepsilon_0} \frac{q}{|\mathbf{r} - (R-h)\hat{\mathbf{z}}|},\tag{4}$$

where \hat{z} denotes the unit vector in direction z.

b) Explain what is the physical meaning of the two terms in Eq. (4), and what role the vectors $\boldsymbol{\rho}_{\pm} = \boldsymbol{r} - (R \pm h)\hat{\boldsymbol{z}}$ have. What is the value of the potential along the plane z = R?

Now instead of the charge at $(R+h)\hat{z}$, we place a charge q > 0 at position $(R+h)\hat{z} + d/2$ and a charge -q at $(R+h)\hat{z} - d/2$, where $d = d\hat{z}$ is a vector for which $d/h \ll 1$ (and d > 0). Note that the grounded conducting half space is still present at $z \leq R$.

c) Make a sketch of the new configuration. Use Eq. (4) to obtain an expression for the scalar potential valid in the region $z \ge R$ under the assumption that $d/\rho_+ \ll 1$. Express your answer in terms of $\boldsymbol{p} = q\boldsymbol{d}$ and $\boldsymbol{\rho}_{\pm}$. Give a physical interpretation of the different terms of the resulting potential.



Another geometry will now be considered. Here a charge q > 0 is placed a distance h > 0 outside the surface of a grounded, conducting sphere of radius R (see figure). We are interested in finding an expression of the scalar potential in the region outside the sphere. To this end, we will once more use the method of images.

d) Derive that an image charge

$$q' = -\frac{R}{R+h}q,\tag{5a}$$

placed a distance

$$z' = \frac{R^2}{R+h},\tag{5b}$$

from the center of the sphere in the radial direction towards the charge q, can be used to solve the electrostatic problem with appropriate boundary conditions. [*Note*: You are not supposed to start from the expressions for q' and z' in showing this. Instead you are asked to derive these results using the boundary conditions.]

e) Describe how the system studied in (b) can be reached as a limit in which the sphere radius $R \to \infty$. Show that in this limit the potential found in (d) reduces to that found in (b).

As above, we will now replace the charge at $(R+h)\hat{z}$ by two opposite charges. One charge q > 0 is placed at $(R+h)\hat{z} + d/2$ and another charge -q at $(R+h)\hat{z} - d/2$ where $d = d\hat{z}$.

f) Use the method of images to find the potential outside the sphere in this case. Determine the charge Q in the monopole term $Q/4\pi\varepsilon_0 r$ in the multipole expansion of the potential. Give a physical interpretation of Q and a qualitative argument for why the obtained sign of Q might be expected.