NTNU
Institutt for fysikk

TFY4240
Solution problem set 5 Autumn 2015

Problem 1.

a) The magnetic field is related to the vector potential by

$$\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{r}). \tag{1}$$

In order to do the partial derivatives more conveniently we first rewrite $\boldsymbol{A}(\boldsymbol{r})$ as

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{r}}{r^3}. \tag{2}$$

In component form we therefore get

$$\begin{aligned} B_i(\boldsymbol{r}) &= \varepsilon_{ijk}\partial_j A_k(\boldsymbol{r}) \\ &= \frac{\mu_0}{4\pi}\varepsilon_{ijk}\partial_j\left(\varepsilon_{klm}\frac{m_l r_m}{r^3}\right) \\ &= \frac{\mu_0}{4\pi}\varepsilon_{kij}\varepsilon_{klm}m_l\,\partial_j\left(\frac{r_m}{r^3}\right) \\ &= \frac{\mu_0}{4\pi}\varepsilon_{kij}\varepsilon_{klm}m_l\left(\frac{1}{r^3}\partial_j r_m + r_m\cdot\frac{(-3)}{r^4}\partial_j r\right) \end{aligned} \tag{3}$$

where we used $\varepsilon_{ijk} = \varepsilon_{kij}$. Now we use

$$\partial_j r_m = \delta_{jm}, \tag{4}$$

$$\partial_j r = \partial_j\sqrt{r_m r_m} = \frac{1}{2r}2r_m\delta_{jm} = \frac{r_j}{r}, \tag{5}$$

which gives, upon also using $\varepsilon_{kij}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$,

$$\begin{aligned} B_i(\boldsymbol{r}) &= \frac{\mu_0}{4\pi}\frac{1}{r^3}\varepsilon_{kij}\varepsilon_{klm}m_l\left(\delta_{jm} - 3\frac{r_j r_m}{r^2}\right) \\ &= \frac{\mu_0}{4\pi}\frac{1}{r^3}(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})m_l\left(\delta_{jm} - 3\frac{r_j r_m}{r^2}\right). \end{aligned} \tag{6}$$

Doing first the summations over l and m, and in the subsequent line the sum over j (using $\delta_{jj} = 3$ and $r_j r_j = r^2$) gives

$$\begin{aligned} B_i(\boldsymbol{r}) &= \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[m_i\left(\delta_{jj} - 3\frac{r_j r_j}{r^2}\right) - m_j\left(\delta_{ji} - 3\frac{r_j r_i}{r^2}\right)\right] \\ &= \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[m_i\left(3 - 3\frac{r^2}{r^2}\right) - m_i + 3\frac{(\boldsymbol{m}\cdot\boldsymbol{r})r_i}{r^2}\right] \\ &= \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3\frac{(\boldsymbol{m}\cdot\boldsymbol{r})r_i}{r^2} - m_i\right]. \end{aligned} \tag{7}$$

Thus, going back to the vector form,

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3\frac{(\boldsymbol{m}\cdot\boldsymbol{r})\boldsymbol{r}}{r^2} - \boldsymbol{m}\right] = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\boldsymbol{m}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{m}]. \tag{8}$$

b) Equation (8) is identical in form to the expression for the electric field of a pure electric dipole under the substitutions $\boldsymbol{E} \to \boldsymbol{B}$, $\boldsymbol{p} \to \boldsymbol{m}$, and $1/\epsilon_0 \to \mu_0$.

Problem 2.

a) The magnetic field from a line element $d\boldsymbol{l}$ is given by Biot-Savart's law,

$$d\boldsymbol{B} = \frac{\mu_0}{4\pi} \frac{I d\boldsymbol{l} \times \hat{\boldsymbol{r}}}{r^2} \tag{9}$$

where $d\boldsymbol{l}$ points along the wire, in the direction of the current. At position O, there is no contribution to the magnetic field from the straight parts of the wire, since the line elements along those parts satisfy $d\boldsymbol{l} \times \hat{\boldsymbol{r}} = 0$. Thus the only contribution comes from the semicircle. There $d\boldsymbol{l} \times \hat{\boldsymbol{r}}$ points out of the paper plane, so therefore the magnetic field does too. Using also that along the semicircle (i) $d\boldsymbol{l}$ is perpendicular to $\hat{\boldsymbol{r}}$, so $|d\boldsymbol{l} \times \hat{\boldsymbol{r}}| = dl$, (ii) $r = R$, the magnitude of the field becomes (the integral goes over the semicircle only)

$$B = \int dB = \frac{\mu_0 I}{4\pi R^2} \underbrace{\int dl}_{\pi R} = \frac{\mu_0 I}{4R} \tag{10}$$

since $\int dl$ is just the semicircle length. As already noted, $\boldsymbol{B}$ points out of the paper plane.

b)

$$|\boldsymbol{B}| = \frac{\mu_0 I}{4R} = 4\pi \cdot 10^{-7}\ \mathrm{N/A^2} \cdot \frac{1}{4} \cdot 10^2\ \mathrm{A/m} \approx 3 \cdot 10^{-5}\ \mathrm{T}. \tag{11}$$

Problem 3.

See Griffiths!