TFY4240
Solution problem set 6 Autumn 2015

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Problem 1.
See Griffiths!

Problem 2.

a) The expression for the current density $\boldsymbol{J}(\boldsymbol{r})$ should satisfy the following properties:

- It should be nonzero only along the infinite wire.
- At each position along the wire, integrating $J(\boldsymbol{r})$ over the cross section of the wire should give the current I. Thus $\int dx \int dy\, \boldsymbol{J}(\boldsymbol{r}) = I\hat{\boldsymbol{z}}$.

The expression that satisfies these properties is

$$\boldsymbol{J}(\boldsymbol{r}) = I\delta(x)\delta(y)\hat{\boldsymbol{z}}. \tag{1}$$

b) From Biot-Savarts law it follows that ($\boldsymbol{R} = \boldsymbol{r}$ - $\boldsymbol{r'}$)

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \mathrm{d}^3 r' \frac{\boldsymbol{J}(\boldsymbol{r'}) \times \hat{\boldsymbol{R}}}{R^2} \tag{2}$$

$$= \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty} \mathrm{d}z' \frac{\hat{\boldsymbol{z}}' \times \hat{\boldsymbol{R}}}{|\boldsymbol{r} - z'\hat{\boldsymbol{z}}|^2} \tag{3}$$

$$= \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}} \tag{4}$$

Here s is the distance from the long wire to the observation point. The last integral was calculated using the method of Example 5.5 in Griffiths.

Alternatively, given the high symmetry of the problem the same result could be obtained very easily from Ampere's law.

c) We first find the flux through the loop:

$$\boldsymbol{\Phi} = \int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} = \frac{\mu_0 I}{2\pi}\int_{s_0}^{s_0+a} \frac{1}{s} a \mathrm{d}s \tag{5}$$

$$= \frac{\mu_0 I a}{2\pi} \ln\left(\frac{s_0 + a}{s_0})\right) \tag{6}$$

By introducing a time-dependent s, we get

$$\boldsymbol{\Phi}(t) = \frac{\mu_0 I a}{2\pi} \ln(\frac{s(t) + a}{s(t)}), \tag{7}$$

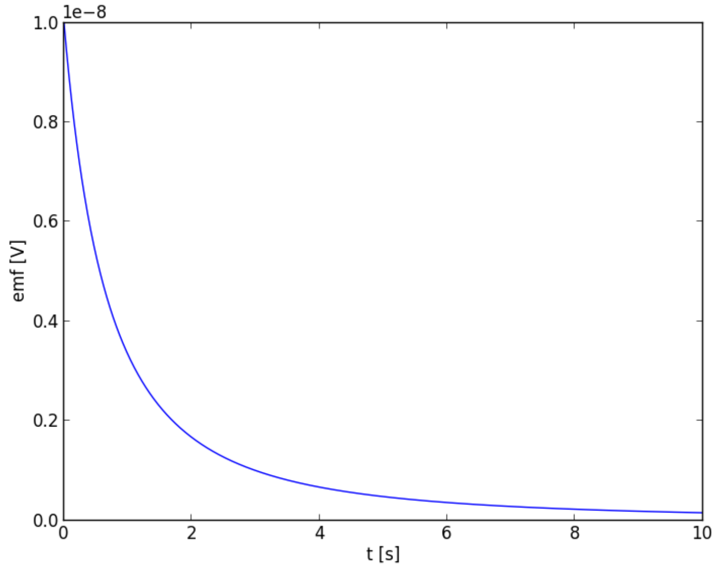

Figure 1: Schematic plot of the emf *vs.* time.

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where in our case $s(t) = s_0 + vt$, since $\boldsymbol{v} = v\hat{\boldsymbol{y}}$.

The generated emf ε can now be calculated from the formula

$$\varepsilon = -\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} = -\frac{\mu_0 I a}{2\pi}\frac{\mathrm{d}}{\mathrm{d}t}\ln\left(\frac{s(t)+a}{s(t)}\right) \tag{8}$$

$$= -\frac{\mu_0 I a}{2\pi}\left[\frac{1}{s(t)+a}\frac{\mathrm{d}s(t)}{\mathrm{d}t} - \frac{1}{s(t)}\frac{\mathrm{d}s(t)}{\mathrm{d}t}\right] \tag{9}$$

$$= \underline{\frac{\mu_0 I v a^2}{2\pi s(t)(s(t)+a)}} \tag{10}$$

where we in the last step have used that $\frac{\mathrm{d}s(t)}{\mathrm{d}t} = v$.

The field points out of the page, so the force on a charge in the segment of the loop closest to the infinite wire is to the right. The force on a charge on the far side of the loop is also to the right, but here the field and therefore the force is weaker. The current in the loop is therefore flowing counterclockwise. (This can also be verified from Lenz's law: Since the flux pointing out of the page is getting weaker as we move the loop away, the induced current will give a flux also pointing out of the page, which means that the current must flow counterclockwise.)

d) See Fig. 1. The emf decays monotonically with time t, going to 0 as $1/t^2$ as $t \to \infty$.

e) Since the magnetic field only depends on the distance to the wire, moving the loop in the z direction will not change the flux through the loop, so $\varepsilon = \frac{\mathrm{d}\Phi}{\mathrm{d}t} = 0$.