TFY4240
Solution problem set 6

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Problem 1.
See handwritten solution further back.

Problem 2.
See handwritten solution further back.

Problem 3.

a) The magnetic field is related to the vector potential by

$$\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{r}). \tag{1}$$

In order to do the partial derivatives more conveniently we first rewrite $\boldsymbol{A}(\boldsymbol{r})$ as

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{r}}{r^3}. \tag{2}$$

In component form we therefore get

$$\begin{aligned} B_i(\boldsymbol{r}) &= \varepsilon_{ijk}\partial_j A_k(\boldsymbol{r}) \\ &= \frac{\mu_0}{4\pi}\varepsilon_{ijk}\partial_j\left(\varepsilon_{klm}\frac{m_l r_m}{r^3}\right) \\ &= \frac{\mu_0}{4\pi}\varepsilon_{kij}\varepsilon_{klm}m_l\,\partial_j\left(\frac{r_m}{r^3}\right) \\ &= \frac{\mu_0}{4\pi}\varepsilon_{kij}\varepsilon_{klm}m_l\left(\frac{1}{r^3}\partial_j r_m + r_m\cdot\frac{(-3)}{r^4}\partial_j r\right) \end{aligned} \tag{3}$$

where we used $\varepsilon_{ijk} = \varepsilon_{kij}$. Now we use

$$\partial_j r_m = \delta_{jm}, \tag{4}$$

$$\partial_j r = \partial_j\sqrt{r_m r_m} = \frac{1}{2r}2r_m\delta_{jm} = \frac{r_j}{r}, \tag{5}$$

which gives, upon also using $\varepsilon_{kij}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$,

$$\begin{aligned} B_i(\boldsymbol{r}) &= \frac{\mu_0}{4\pi}\frac{1}{r^3}\varepsilon_{kij}\varepsilon_{klm}m_l\left(\delta_{jm} - 3\frac{r_j r_m}{r^2}\right) \\ &= \frac{\mu_0}{4\pi}\frac{1}{r^3}(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})m_l\left(\delta_{jm} - 3\frac{r_j r_m}{r^2}\right). \end{aligned} \tag{6}$$

Doing first the summations over l and m, and in the subsequent line the sum over j (using $\delta_{jj} = 3$ and $r_j r_j = r^2$) gives

$$\begin{aligned} B_i(\boldsymbol{r}) &= \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[m_i\left(\delta_{jj} - 3\frac{r_j r_j}{r^2}\right) - m_j\left(\delta_{ji} - 3\frac{r_j r_i}{r^2}\right)\right] \\ &= \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[m_i\left(3 - 3\frac{r^2}{r^2}\right) - m_i + 3\frac{(\boldsymbol{m}\cdot\boldsymbol{r})r_i}{r^2}\right] \\ &= \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3\frac{(\boldsymbol{m}\cdot\boldsymbol{r})r_i}{r^2} - m_i\right]. \end{aligned} \tag{7}$$

Thus, going back to the vector form,

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3\frac{(\boldsymbol{m}\cdot\boldsymbol{r})\boldsymbol{r}}{r^2} - \boldsymbol{m}\right] = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\boldsymbol{m}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{m}]. \tag{8}$$

b) Equation (8) is identical in form to the expression for the electric field of a pure electric dipole under the substitutions $\boldsymbol{E} \to \boldsymbol{B}$, $\boldsymbol{p} \to \boldsymbol{m}$, and $1/\epsilon_0 \to \mu_0$.

Problem 4.
See handwritten solution further back.

Problem 5.
See Griffiths.

Problem 1

(a) The Poisson eq. is $\nabla^2 V = -\frac{\rho_f}{\epsilon}$

The free charge consists of point particles at positions $\vec{r}_i$ with charge Q_i

$$\Rightarrow \quad \rho_f(\vec{r}) = \sum_i Q_i \, \delta(\vec{r} - \vec{r}_i)$$

$$\Rightarrow \quad V(\vec{r}) = \frac{1}{4\pi\epsilon} \sum_i \frac{Q_i}{|\vec{r} - \vec{r}_i|} \qquad \text{(general form of solution)}$$

In this problem, all charges are on the z axis, i.e. $\vec{r}_i = (0, 0, z_i)$

$$\Rightarrow \quad V(\vec{r}) = \frac{1}{4\pi\epsilon} \sum_i \frac{Q_i}{\sqrt{x^2 + y^2 + (z - z_i)^2}}$$

$$\Rightarrow \quad \partial_z V(\vec{r})\Big|_{z=0} = \frac{1}{4\pi\epsilon}\left(-\frac{1}{2}\right) \sum_i \frac{Q_i \cdot 2(z - z_i) \cdot 1}{[x^2 + y^2 + (z - z_i)^2]^{3/2}}\Bigg|_{z=0}$$

$$= \frac{1}{4\pi\epsilon} \sum_i \frac{Q_i z_i}{[x^2 + y^2 + z_i^2]^{3/2}} \quad (*) \quad \text{(needed later)}$$

$V_1(\vec{r})$: $\epsilon = \epsilon_1$, image point charge q_1 at $(0, 0, d)$

$$\Rightarrow \quad V_1(\vec{r}) = \frac{1}{4\pi\epsilon_1} \frac{q_1}{\sqrt{x^2 + y^2 + (z - d)^2}} \qquad \text{(valid for } z<0\text{)}$$

$V_2(\vec{r})$: $\epsilon = \epsilon_2$, point charge q at $(0, 0, d)$, image charge q_2 at $(0, 0, -d)$

$$V_2(\vec{r}) = \frac{1}{4\pi\epsilon_2}\left[\frac{q}{\sqrt{x^2 + y^2 + (z - d)^2}} + \frac{q_2}{\sqrt{x^2 + y^2 + (z + d)^2}}\right] \qquad \text{(valid for } z>0\text{)}$$

Boundary / matching conditions at the interface $(z=0)$:

$$V_1(\vec{r}) = V_2(\vec{r}) \qquad (1)$$

$$\epsilon_2 \partial_z V_2(\vec{r}) - \epsilon_1 \partial_z V_1(\vec{r}) = -\sigma_f = 0 \qquad (2)$$

$$(1) \Rightarrow \frac{1}{\cancel{4\pi}\epsilon_1} \frac{q_1}{\cancel{\sqrt{x^2+y^2+d^2}}} = \frac{1}{\cancel{4\pi}\epsilon_2} \frac{q+q_2}{\cancel{\sqrt{x^2+y^2+d^2}}}$$

$$\Rightarrow \frac{q_1}{\epsilon_1} = \frac{q+q_2}{\epsilon_2} \qquad (**)$$

$$(2) \overset{(*)}{\Rightarrow} \cancel{\epsilon_2} \frac{1}{\cancel{4\pi\epsilon_2}} \frac{q\cdot \cancel{d} + q_2(-\cancel{d})}{\cancel{[x^2+y^2+d^2]^{3/2}}} - \cancel{\epsilon_1} \frac{1}{\cancel{4\pi\epsilon_1}} \frac{q_1 \cancel{d}}{\cancel{[x^2+y^2+d^2]^{3/2}}} = 0$$

$$\Rightarrow q - q_2 = q_1$$

Insert into $(**)$ $\Rightarrow \dfrac{q-q_2}{\epsilon_1} = \dfrac{q+q_2}{\epsilon_2}$

$$\Rightarrow: \quad q_2\left(\frac{1}{\epsilon_2} + \frac{1}{\epsilon_1}\right) = q\left(\frac{1}{\epsilon_1} - \frac{1}{\epsilon_2}\right)$$

$$\Rightarrow \quad \underline{\underline{q_2}} = q \frac{\frac{1}{\epsilon_1} - \frac{1}{\epsilon_2}}{\frac{1}{\epsilon_2} + \frac{1}{\epsilon_1}} = q \frac{\epsilon_2 - \epsilon_1}{\epsilon_1 + \epsilon_2} = \underline{\underline{-\frac{K_1 - K_2}{K_1 + K_2} q}}$$

$$\underline{\underline{q_1}} = q - q_2 = \left[1 + \frac{K_1 - K_2}{K_1 + K_2}\right] q = \underline{\underline{\frac{2K_1}{K_1 + K_2} q}}$$

(b) In region 2 the potential $V(\vec{r})$ equals

$$V_2(\vec{r}) = \frac{1}{4\pi\epsilon_2}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{q_2}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

The contribution to the electric field $\vec{E} = -\nabla V$ from the first term in V_2 is due to the point charge q. Thus this term must be excluded when calculating the force $\vec{F}$ on q (since q doesn't act with a force on itself). Thus

$$\vec{F} = q(-\nabla)\frac{1}{4\pi\epsilon_2}\frac{q_2}{\sqrt{x^2+y^2+(z+d)^2}}\Bigg|_{(x=0,y=0,z=d)}$$

$$= -\frac{qq_2}{4\pi\epsilon_2}\left(-\frac{1}{2}\right)\frac{2}{[x^2+y^2+(z+d)^2]^{3/2}}\left[x\hat{x}+y\hat{y}+(z+d)\hat{z}\right]\Bigg|_{(0,0,d)}$$

$$= \frac{qq_2}{4\pi\epsilon_2}\frac{1}{[(2d)^2]^{3/2}}\cdot 2d\,\hat{z} = \underline{\underline{-\frac{K_1-K_2}{K_1+K_2}\frac{q^2}{4\pi\epsilon_2(2d)^2}\hat{z}}}$$

Comments:

— The force points towards the interface for $K_1 > K_2$ and away from the interface for $K_1 < K_2$. Let us try to understand this. The force on q is due to bound charge. There is volume bound charge at the position of q, but this gives zero net force on q. Thus the force is due to surface bound charge at the interface. For concreteness, let us assume that q is positive. Then positive bound charge is repelled by q and negative bound charge is attracted by q. Therefore there is a positive surface bound charge on the top (medium 2) side of the interface and a negative surface bound charge on the bottom (medium 1) side of the interface. The sign of the net surface bound charge is determined by which medium has the greatest ability to polarize (i.e. biggest K). If $K_1 > K_2$ the net surface bound charge is negative, giving an attractive force, while if $K_1 < K_2$ the net surface bound charge is positive, giving a repulsive force. (If q is negative, just change positive $\Leftrightarrow$ negative in these arguments; the direction of the force is unchanged).

- The force is zero if $K_1 = K_2$, since then there is really just a single medium and thus no interface.
- Also note that when $K_2 = 1$ (i.e. medium 2 is vacuum, $\epsilon_2 = \epsilon_0$) and $K_1 \to \infty$, the results reproduce those for the "classic image problem" investigated earlier in the course (a point charge in vacuum above a grounded conducting plane).

Problem 2

(a) $$\nabla \cdot \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \nabla \cdot \left(\frac{1}{R} \vec{j}(\vec{r}') \right)$$ where

$$\nabla \cdot \left(\frac{1}{R} \vec{j}(\vec{r}') \right) = \frac{1}{R} \underbrace{\nabla \cdot \vec{j}(\vec{r}')}_{=0} + \vec{j}(\vec{r}') \cdot \underbrace{\nabla \frac{1}{R}}_{= -\nabla' \frac{1}{R}} = -\vec{j}(\vec{r}') \cdot \nabla' \frac{1}{R}$$

Using identity (5) in Griffiths,

$$\vec{j}(\vec{r}') \cdot \nabla' \frac{1}{R} = \frac{1}{R} \underbrace{\nabla' \cdot \vec{j}(\vec{r}')}_{\substack{=0 \text{ (steady} \\ \text{current condition)}}} - \nabla' \cdot \left(\frac{1}{R} \vec{j}(\vec{r}') \right)$$

$$\Rightarrow \nabla \cdot \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \nabla' \cdot \left(\frac{1}{R} \vec{j}(\vec{r}') \right) = \frac{\mu_0}{4\pi} \int d\vec{a} \cdot \left(\frac{1}{R} \vec{j}(\vec{r}') \right)$$

where we used the divergence theorem. The current is zero on the boundary, since the volume integral is over a region big enough that the current is entirely inside (we assume a localized current distribution)

$$\Rightarrow \nabla \cdot \vec{A}(\vec{r}) = 0 .$$ QED.

(b) $$\vec{B}(\vec{r}) = \nabla \times \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \nabla \times \left(\frac{1}{R} \vec{j}(\vec{r}') \right)$$

$$= \frac{\mu_0}{4\pi} \int d^3r' \left[\frac{1}{R} \underbrace{\nabla \times \vec{j}(\vec{r}')}_{=0} - \vec{j}(\vec{r}') \times \underbrace{\nabla \frac{1}{R}}_{-\hat{R}/R^2} \right]$$

$$= \frac{\mu_0}{4\pi} \int d^3r' \, \frac{\vec{j}(\vec{r}') \times \hat{R}}{R^2}$$ QED

Problem 4 In the lectures we showed that

$$\vec{B}_{M,\text{inside}} = \frac{2\mu_0}{3}\vec{M}$$

$$\vec{B}_{M,\text{outside}} = \nabla \times \vec{A}_{M,\text{outside}}, \text{ where}$$

$$\vec{A}_{M,\text{outside}} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}, \text{ i.e. a dipole field } \left(\vec{m} = \frac{4\pi}{3}R^3\vec{M}\right)$$

(a) $\vec{B} = \vec{B}_0 + \vec{B}_M$

Inside: $\vec{B}_M = \frac{2\mu_0}{3}\vec{M}$

$$\vec{M} = \chi_m \vec{H} = \frac{\chi_m}{\mu}\vec{B} = \frac{K_m - 1}{\mu_0 K_m}\vec{B} = \frac{1}{\mu_0}\left(1 - \frac{1}{K_m}\right)\vec{B}$$

$$\Rightarrow \vec{B} = \vec{B}_0 + \frac{2\cancel{\mu_0}}{3}\frac{1}{\cancel{\mu_0}}\left(1 - \frac{1}{K_m}\right)\vec{B}$$

$$\Rightarrow \vec{B} = \frac{\vec{B}_0}{1 - \frac{2}{3}\left(1 - \frac{1}{K_m}\right)} = \frac{\vec{B}_0}{\frac{1}{3} + \frac{2}{3K_m}} = \underline{\underline{\frac{3K_m}{K_m + 2}\vec{B}_0 = \frac{3}{1 + 2/K_m}\vec{B}_0}}$$

magnetic field inside

(b)

Condition	K_m	χ_m	μ
$\lvert\vec{B}\rvert > \lvert\vec{B}_0\rvert$	>1	>0	$>\mu_0$
$\lvert\vec{B}\rvert = \lvert\vec{B}_0\rvert$	1	0	μ_0
$\lvert\vec{B}\rvert < \lvert\vec{B}_0\rvert$	<1	<0	$<\mu_0$
$\lvert\vec{B}\rvert = 0$	0	-1	0

(Used that $\mu = (1 + \chi)\mu_0 = K_m\mu_0$)

(c) Magnetization: $\vec{M} = \frac{K_m - 1}{\mu_0 K_m}\vec{B} = \underline{\underline{\frac{3}{\mu_0}\frac{K_m - 1}{K_m + 2}\vec{B}_0}}$

$$\vec{m} = \frac{4\pi}{3}R^3\vec{M} = \underline{\underline{4\pi R^3 \frac{1}{\mu_0}\frac{K_m - 1}{K_m + 2}\vec{B}_0}}$$

(d) Outside: $\vec{B} = \vec{B}_0 + \vec{B}_{M,\text{outside}}$