TFY4240

Solution problem set 7 Autumn 2015

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Problem 1.

a) The flux is given by

$$\phi(x) = \int \boldsymbol{B} \cdot d\boldsymbol{a} = \ell \int_0^x ds B_n = \ell x B \sin(\pi/4) = \frac{1}{\sqrt{2}} B\ell x. \tag{1}$$

The magnitude of the induced emf is

$$|\varepsilon| = \left|\frac{d\phi}{dt}\right| = \left|\frac{d\phi}{dx}\frac{dx}{dt}\right| = \frac{1}{\sqrt{2}} B\ell v. \tag{2}$$

The current is therefore

$$I = \frac{|\varepsilon|}{R} = \frac{B\ell v}{\sqrt{2}R}. \tag{3}$$

The current flows clockwise (when the circuit is viewed from above). This can be seen from the magnetic Lorentz force on positive charges in the rod moving with velocity $\boldsymbol{v}$: $\boldsymbol{F}_\mathrm{q} = q\boldsymbol{v} \times \boldsymbol{B}$, which points in the $+z$ direction. Alternatively, it can be deduced from Lenz's law: The flux through the loop is increasing, so the induced current will try to oppose this by setting up a magnetic field that points downwards inside the circuit, which (using the right-hand rule for your thumb and curled fingers) implies that the current flows clockwise.

b) The force on the rod is given by

$$\boldsymbol{F}_\mathrm{rod} = I\boldsymbol{\ell} \times \boldsymbol{B}, \tag{4}$$

where $\boldsymbol{\ell}$ points in the $+z$ direction (the direction of the current through the rod). The angle between $\boldsymbol{\ell}$ and $\boldsymbol{B}$ is 90°, so the magnitude of the force is

$$|\boldsymbol{F}_\mathrm{rod}| = I\ell B = \frac{B^2\ell^2 v}{\sqrt{2}R}. \tag{5}$$

From the right-hand rule, the direction of $\boldsymbol{F}_\mathrm{rod}$ is in the xy plane, at a 135° angle with the x axis. Thus the force components are

$$F_\mathrm{rod,y} = -F_\mathrm{rod,x} = \frac{B^2\ell^2 v}{2R}. \tag{6}$$

As the rod should move in the x direction with constant velocity v, a force of magnitude $|F_\mathrm{rod,x}|$ pointing in the positive x direction needs to be applied to the rod by an outside agent in order to balance the negative x component of $\boldsymbol{F}_{\text{rod}}$ (Newton's 1st law). The mechanical power provided by the outside agent is therefore

$$P_{\text{mech}} = |F_{\text{rod,x}}|v = \frac{B^2\ell^2v^2}{2R}. \tag{7}$$

This power is dissipated in the resistor, as the ohmic power loss is

$$P_{\text{electric}} = RI^2 = \frac{B^2\ell^2v^2}{2R} = P_{\text{mech}}. \tag{8}$$

c) The vertical component of the force is given by $F_{\text{rod,y}}$ in (6). The rod levitates when this component is larger than the gravitational force mg, i.e. when

$$B > \sqrt{\frac{2mgR}{\ell^2 v}}. \tag{9}$$

d) This problem is essentially the same as Example 5.8 in Griffiths, so I refer to the discussion there. In our case, the magnetic field above the conducting plane will point in the negative x direction and have a constant magnitude $B = \mu_0 K/2$.

e) When the direction of the current I is opposite to that of $\boldsymbol{K}$, the force $I\boldsymbol{\ell} \times \boldsymbol{B}$ will be upwards. Levitation requires $I\ell B > mg$, i.e.

$$I > \frac{mg}{B\ell} = \frac{9.8\text{m/s}^2}{2.0 \text{ T}} \cdot \frac{1000 \text{ kg}}{1 \text{ m}} = 4.9 \cdot 10^3 \text{ A}. \tag{10}$$

Problem 2.

a) To calculate $\boldsymbol{\nabla} \cdot \overleftrightarrow{T}$ one has to keep in mind that $\boldsymbol{\nabla} = \hat{\boldsymbol{x}}_i \partial_i$ operates on what appears to its right. The effect of $\boldsymbol{\nabla}$ on $\overleftrightarrow{T} = T_{ij}\hat{\boldsymbol{x}}_i\hat{\boldsymbol{x}}_j$ is limited to the components T_{ij} since the Cartesian unit vectors $\hat{\boldsymbol{x}}_i$ are independent of $\boldsymbol{r}$. This gives

$$\begin{aligned} \boldsymbol{\nabla} \cdot \overleftrightarrow{T} &= [\hat{\boldsymbol{x}}_i\partial_i] \cdot [T_{jk}\, \hat{\boldsymbol{x}}_j\hat{\boldsymbol{x}}_k] \\ &= [\partial_i T_{jk}]\ \hat{\boldsymbol{x}}_i \cdot (\hat{\boldsymbol{x}}_j\hat{\boldsymbol{x}}_k) \\ &= [\partial_i T_{jk}]\ \underbrace{(\hat{\boldsymbol{x}}_i \cdot \hat{\boldsymbol{x}}_j)}_{\delta_{ij}} \hat{\boldsymbol{x}}_k \\ &= [\partial_i T_{ik}]\ \hat{\boldsymbol{x}}_k. \end{aligned} \tag{11}$$

Hence, component i of the vector $\boldsymbol{\nabla} \cdot \overleftrightarrow{T}$ becomes

$$\left[\boldsymbol{\nabla} \cdot \overleftrightarrow{T}\right]_i = \partial_j T_{ji}. \tag{12}$$

b) We get

$$\begin{aligned} \boldsymbol{v} \times \overleftrightarrow{T} &= [v_k\hat{\boldsymbol{x}}_k] \times [T_{ij}\, \hat{\boldsymbol{x}}_i\hat{\boldsymbol{x}}_j] \\ &= [v_k T_{ij}]\ \hat{\boldsymbol{x}}_k \times (\hat{\boldsymbol{x}}_i\hat{\boldsymbol{x}}_j) \\ &= [v_k T_{ij}]\ \underbrace{(\hat{\boldsymbol{x}}_k \times \hat{\boldsymbol{x}}_i)}_{\varepsilon_{ki\ell}\hat{\boldsymbol{x}}_\ell} \hat{\boldsymbol{x}}_j \\ &= \varepsilon_{ki\ell} v_k T_{ij} \hat{\boldsymbol{x}}_\ell \hat{\boldsymbol{x}}_j \\ &= \varepsilon_{k\ell i} v_k T_{\ell j} \hat{\boldsymbol{x}}_i \hat{\boldsymbol{x}}_j \end{aligned} \tag{13}$$

where in the last step we renamed the dummy indices i and ℓ, in order to read off the ij component of $\boldsymbol{v} \times \overleftrightarrow{T}$ as what multiplies $\hat{\boldsymbol{x}}_i\hat{\boldsymbol{x}}_j$. This gives

$$(\boldsymbol{v} \times \overleftrightarrow{T})_{ij} = \varepsilon_{ik\ell} v_k T_{\ell j} \tag{14}$$

(here we also used $\varepsilon_{k\ell i} = \varepsilon_{ik\ell}$ which follows from the invariance of the Levi-Civita tensor under cyclic permutations of indices).

Similar manipulations give

$$\begin{aligned}
\overleftrightarrow{T} \times \boldsymbol{v} &= [T_{ij}\hat{\boldsymbol{x}}_i\hat{\boldsymbol{x}}_j] \times [v_k\hat{\boldsymbol{x}}_k] \\
&= [T_{ij}v_k](\hat{\boldsymbol{x}}_i\hat{\boldsymbol{x}}_j) \times \hat{\boldsymbol{x}}_k \\
&= [T_{ij}v_k]\hat{\boldsymbol{x}}_i \underbrace{(\hat{\boldsymbol{x}}_j \times \hat{\boldsymbol{x}}_k)}_{\varepsilon_{jk\ell}\hat{\boldsymbol{x}}_\ell} \\
&= \varepsilon_{jk\ell} T_{ij} v_k \hat{\boldsymbol{x}}_i\hat{\boldsymbol{x}}_\ell \\
&= \varepsilon_{\ell kj} T_{i\ell} v_k \hat{\boldsymbol{x}}_i\hat{\boldsymbol{x}}_j,
\end{aligned} \tag{15}$$

so that the ij component of $\overleftrightarrow{T} \times \boldsymbol{v}$ is

$$(\overleftrightarrow{T} \times \boldsymbol{v})_{ij} = \varepsilon_{jk\ell} T_{ik} v_\ell. \tag{16}$$

As an example of the relevance of this cross product between a rank-2 tensor and a vector, we note that the angular momentum current density (which was mentioned in the lectures in connection with a very brief discussion of conservation of angular momentum) is given by $\overleftrightarrow{T} \times \boldsymbol{r}$, where $\overleftrightarrow{T}$ is here the Maxwell stress tensor and $\boldsymbol{r}$ is the position vector.