# TFY4240
# Problem set 7

NTNU
Institutt for fysikk

**Problem 1.**
In the lectures I stated without proof that the integral of the magnetization $\boldsymbol{M}$ over the total volume $\Omega$ of a sample of magnetic material equals the total magnetic moment $\boldsymbol{m}$ of the sample:

$$\boldsymbol{m} = \int_\Omega d^3r \, \boldsymbol{M}(\boldsymbol{r}). \tag{1}$$

The purpose of this problem is to prove (1).

**a)** By applying the divergence theorem $\int_\Omega d^3r \, \nabla \cdot \boldsymbol{v} = \int_a d\boldsymbol{a} \cdot \boldsymbol{v}$ to the case $\boldsymbol{v}(\boldsymbol{r}) = f(\boldsymbol{r})\boldsymbol{c}$, where $\boldsymbol{c}$ is a constant vector, show that

$$\int_\Omega d^3r \, \nabla f = \int_a d\boldsymbol{a} f. \tag{2}$$

This result will be used later.

We start from the definition of the magnetic moment $\boldsymbol{m}$ due to a current density $\boldsymbol{j}$:

$$\boldsymbol{m} = \frac{1}{2} \int d^3r \, \boldsymbol{r} \times \boldsymbol{j}(\boldsymbol{r}). \tag{3}$$

Applying this to the bound currents of a sample of material, and separating the contributions from the volume and surface of the sample, with associated bound volume current density $\nabla \times \boldsymbol{M}$ and bound surface current density $\boldsymbol{M} \times \hat{\boldsymbol{n}}$, gives

$$\boldsymbol{m} = \frac{1}{2} \int_\Omega d^3r \, \boldsymbol{r} \times (\nabla \times \boldsymbol{M}) + \frac{1}{2} \int_a da \, \boldsymbol{r} \times (\boldsymbol{M} \times \hat{\boldsymbol{n}}). \tag{4}$$

**b)** Consider the $i$'th component $m_i$. Express the components of the cross products using the Levi-Civita symbol, and use the identity $\epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ to show that

$$m_i = \frac{1}{2} \int_\Omega d^3r \, \left[ r_j \partial_i M_j - r_j \partial_j M_i \right] + \frac{1}{2} \int_a da \left[ r_j M_i \hat{n}_j - r_j M_j \hat{n}_i \right]. \tag{5}$$

**c)** Use the identity $\partial_k(AB) = A\,\partial_k B + B\,\partial_k A$ to rewrite each term in the volume integral. Next, show that the resulting volume integrals of the total derivative terms $\partial_k(AB)$ cancel the surface integrals in (5) (to show this cancellation, invoke the result (2) componentwise). Finally, simplify the remaining volume integral terms to show that $m_i = \int_\Omega d^3r \, M_i$, which concludes the proof.

**Problem 2.**
(This is essentially Problem 6.12 in Griffiths, slightly rewritten and with a hint.) An infinitely long cylinder, of radius $R$, has a "frozen-in" magnetization $\boldsymbol{M}$ parallel to the cylinder axis (taken to be the $\hat{\boldsymbol{z}}$ axis) given by

$$\boldsymbol{M} = ks\hat{\boldsymbol{z}}, \tag{6}$$

where $k$ is a constant and $s$ is the distance from the $z$ axis. There is no free current anywhere. Find the magnetic field inside and outside the cylinder by two different methods:

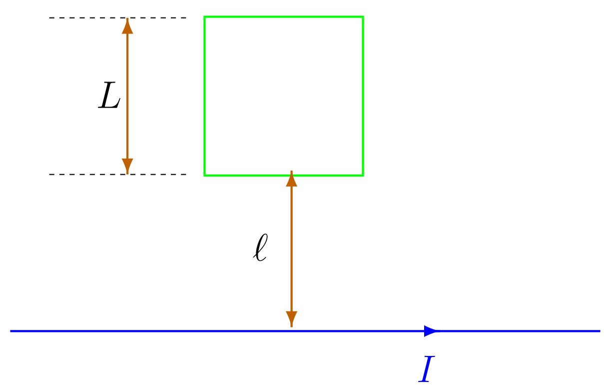
**a)** First calculate the bound current densities, and then calculate the total magnetic field they produce. [Hint: After you have found the bound current densities, argue that the bound currents in this problem can be regarded as making up a continuous set of concentric, infinitely long solenoids with different radii. Then use the known results for the magnetic field due to a solenoid (see e.g. Ex. 5.9 in Griffiths) to calculate the magnetic field from the bound currents.]

**b)** First find $\boldsymbol{H}$, and then find $\boldsymbol{B}$.

You should find the second method considerably easier than the first.

**Problem 3.**
Go through Examples 7.2, 7.4, and 7.9 in Griffiths.

**Problem 4.**
(This is a slightly rewritten version of problem 7.8 in Griffiths.) A square loop of wire with side length $L$ lies on a table, a distance $\ell$ from an infinitely long straight wire which carries a constant current $I$.




**a)** Find the magnetic flux through the loop.

**b)** What is the emf generated if the loop is pulled at constant speed $v$ directly away from the wire? What is the direction of the induced current?

**c)** Same questions as in (b), except that the loop is now pulled to the right, i.e. the velocity is parallel to the wire.

**Problem 5.**

Let $\boldsymbol{v}$ be a general vector and let $\overleftrightarrow{T}$ be a general tensor of rank 2. In terms of the basis vectors $\hat{\boldsymbol{x}}_i$ ($i = 1, 2, 3$), one can write

$$\begin{aligned} \boldsymbol{v} &= v_i \hat{\boldsymbol{x}}_i, \\ \overleftrightarrow{T} &= T_{ij} \hat{\boldsymbol{x}}_i \hat{\boldsymbol{x}}_j \end{aligned}$$

where $\hat{\boldsymbol{x}}_i \hat{\boldsymbol{x}}_j \equiv \hat{\boldsymbol{x}}_i \otimes \hat{\boldsymbol{x}}_j$.

**a)** Determine the components of $\boldsymbol{\nabla} \cdot \overleftrightarrow{T}$.

**b)** In addition to the dot product ($\cdot$) between a vector and a rank-2 tensor, one can also define a cross product ($\times$) between a vector and a rank-2 tensor. Like the dot product, the cross product is noncommutative. Unlike the dot product, whose result is a vector, the cross product gives another rank-2 tensor. The cross products $\overleftrightarrow{T} \times \boldsymbol{v}$ and $\boldsymbol{v} \times \overleftrightarrow{T}$ can be evaluated term by term using the definitions

$$\begin{aligned} (\hat{\boldsymbol{x}}_i \hat{\boldsymbol{x}}_j) \times \hat{\boldsymbol{x}}_k &\equiv \hat{\boldsymbol{x}}_i (\hat{\boldsymbol{x}}_j \times \hat{\boldsymbol{x}}_k), \\ \hat{\boldsymbol{x}}_k \times (\hat{\boldsymbol{x}}_i \hat{\boldsymbol{x}}_j) &\equiv (\hat{\boldsymbol{x}}_k \times \hat{\boldsymbol{x}}_i) \hat{\boldsymbol{x}}_j. \end{aligned}$$

Use these to find expressions for the components of $\overleftrightarrow{T} \times \boldsymbol{v}$ and $\boldsymbol{v} \times \overleftrightarrow{T}$. (The result $\hat{\boldsymbol{x}}_i \times \hat{\boldsymbol{x}}_j = \varepsilon_{ijk} \hat{\boldsymbol{x}}_k$ may be useful.)