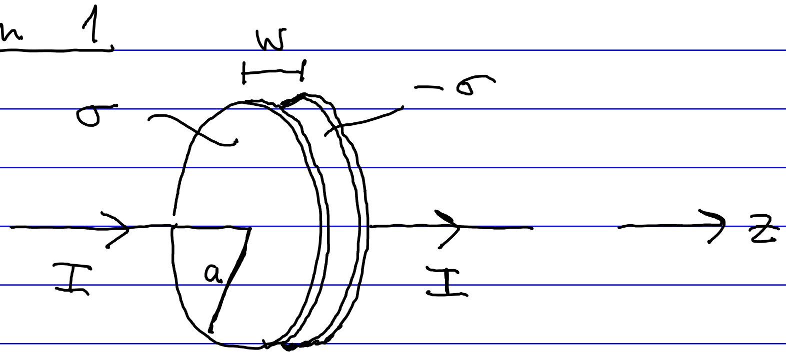
Tutorial 8, TFY4240 2015, solution

Problem 1.

(a)



The electric field in the gap is normal to the plates, pointing from the positively to the negatively charged plate, and has magnitude

$$E = \frac{\sigma}{\epsilon_0}$$

where $\pm\sigma$ is the surface charge density the two plates (see Ex. 2.5 and Ex. 2.10 in Griffiths; this expression neglects the "fringing field" around the edges; see Sec. 4.4.4).

The charge $\pm Q$ on the plates is given by $Q = \sigma A$ with $A = \pi a^2$. With $Q = 0$ at time $t = 0$ and a time-independent current I we get $Q = It$
$\Rightarrow \sigma = It/A$

$$\Rightarrow \quad \underline{E = \frac{It}{\epsilon_0 A}}$$

To find $\vec{B}$ we consider a circular Amperian loop C' of radius s inside the gap, oriented $\parallel$ to the plates and with a center on the axis passing through both plate centers. We take the circular disk of radius s bounded by C' as the surface S' in Stokes's theorem.

Then the Ampere-Maxwell law gives

$$\oint_{C'} \vec{B} \cdot d\vec{l} = \mu_0 \int_{S'} \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{a}$$

Since there is no current in the gap, $\vec{J} = 0$ on S'.
Furthermore,

$$\frac{\partial \vec{E}}{\partial t} = \frac{I}{\epsilon_0 A} \qquad \text{(a constant)}$$

$$\Rightarrow \mu_0 \epsilon_0 \int \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a} = \mu_0 \cancel{\epsilon_0} \frac{I}{\cancel{\epsilon_0} A} \cdot \pi s^2 = \mu_0 I \left(\frac{s}{a}\right)^2$$

By symmetry, $\vec{B}$ will be tangential on C' and have a magnitude that only depends on s. Thus

$$\oint_{C'} \vec{B} \cdot d\vec{l} = B \cdot 2\pi s$$

$$\Rightarrow B = \frac{\mu_0 I}{2\pi s} \left(\frac{s}{a}\right)^2 = \underline{\underline{\frac{\mu_0 I s}{2A}}}$$

The direction of $\vec{B}$ is clockwise around C' when viewed from the positive plate (this follows from a right-hand rule relating the positive circulation direction around C' to the positive direction of S').

(I called the surface S' to avoid confusion with the Poynting vector $\vec{S}$ and its magnitude S.)

(b) The energy density u_{EM} in the gap is

$$u_{EM} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

$$= \frac{1}{2}\left(\epsilon_0 \left(\frac{It}{\epsilon_0 A}\right)^2 + \frac{1}{\mu_0}\left(\frac{\mu_0 I s}{2A}\right)^2\right)$$

$$= \underline{\underline{\frac{I^2}{2A^2}\left(\frac{t^2}{\epsilon_0} + \frac{\mu_0 s^2}{4}\right)}}$$

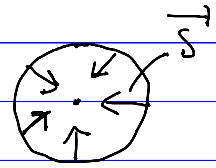
The Poynting vector $\vec{S}$ in the gap is

$$\vec{S} = \frac{1}{\mu_0}\left(\vec{E} \times \vec{B}\right)$$

Introducing cylindrical coordinates (s, ϕ, z) with the z axis going through the plate centers, we find that $\vec{S}$ points in the negative radial direction (ie towards the z axis) and has magnitude

$$S = \frac{1}{\mu_0} E B \sin\frac{\pi}{2} = \frac{1}{\cancel{\mu_0}} \frac{It}{\epsilon_0 A} \frac{\cancel{\mu_0} I s}{2A} = \frac{I^2 s t}{2\epsilon_0 A^2}$$

i.e. $$\underline{\underline{\vec{S} = -\frac{I^2 s t}{2\epsilon_0 A^2}\,\hat{e}_s}} \quad \left(\equiv S_s \hat{e}_s\right)$$



(c) Since there are no charges in the gap, $u_{mech} = 0 \Rightarrow \frac{\partial u_{mech}}{\partial t} = 0$. From (b) we find

$$\frac{\partial u_{EM}}{\partial t} = \underline{\underline{\frac{I^2 t}{A^2 \epsilon_0}}}$$

looking up expression for the divergence in cylindrical coordinates

and $$-\nabla\cdot\vec{S} = -\frac{1}{s}\frac{\partial}{\partial s}\left(s S_s\right) = \frac{I^2 t}{2\epsilon_0 A^2}\frac{1}{s}\frac{\partial}{\partial s}(s^2) = \underline{\underline{\frac{I^2 t}{\epsilon_0 A^2}}} = \frac{\partial u_{EM}}{\partial t}$$

Q.E.D.

(d) Since there are no charges in the gap, $\frac{dW}{dt} = 0$.
Furthermore,

$$\frac{d}{dt}\int_{\Omega} u_{EM}\, d\tau = \frac{d}{dt}\int_{\Omega} \frac{I^2}{2A^2}\left(\frac{t^2}{\epsilon_0} + \underbrace{\frac{\mu_0 s^2}{4}}_{\text{time-indep} \Rightarrow \frac{d}{dt} \to 0}\right) d\tau$$

$$= \frac{I^2}{2A^2}\frac{1}{\epsilon_0}\cdot 2t \cdot w \cdot \pi b^2 = \underline{\underline{\frac{I^2 \pi b^2 w t}{\epsilon_0 A^2}}}$$

Integrating $\vec{S}$ over the boundary $\partial\Omega$ of Ω, there is no contribution from the two end faces since $\vec{S}\cdot d\vec{a} = 0$ there. Thus the only contribution comes from the curved part where $\hat{n}$ points in the opposite direction of $\vec{S}$, and where S is constant ($s=b$)

$$\Rightarrow -\oint_{\partial\Omega} \vec{S}\cdot d\vec{a} = -(-1)\frac{I^2 b t}{2\epsilon_0 A^2}\cdot w \cdot 2\pi b = \underline{\underline{\frac{I^2 \pi b^2 w t}{\epsilon_0 A^2}}}$$

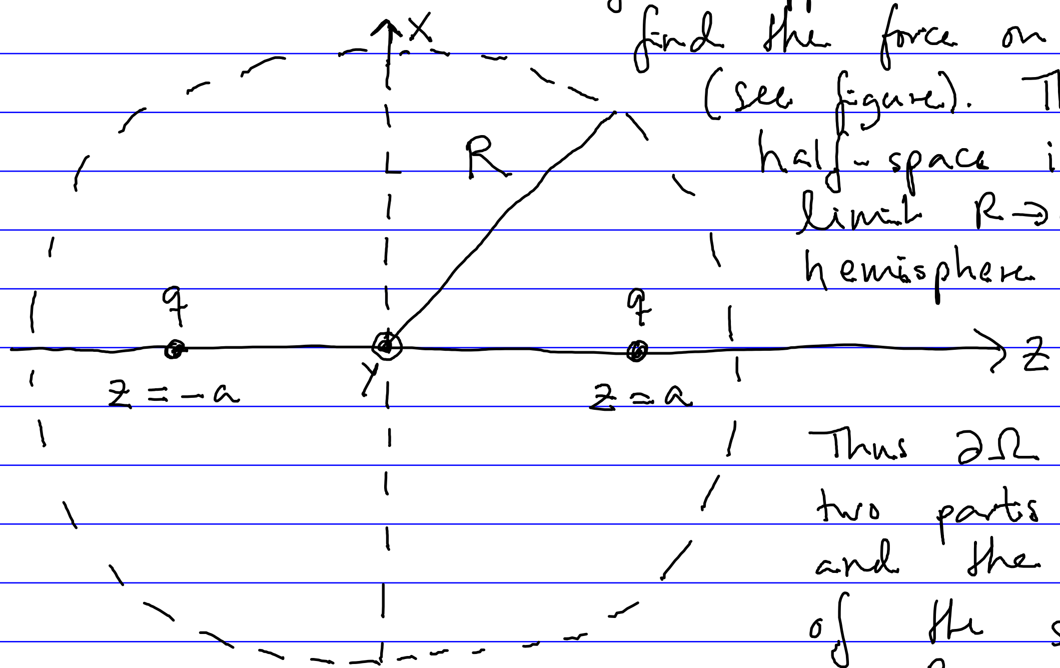
$$= \frac{d}{dt}\int_{\Omega} u_{EM}\, d\tau \qquad \text{Q.E.D.}$$

Problem 2 (problem 8.4 in Griffiths)

(a) Of course, in this problem the force is already known; it is just given by Coulomb's law. But the point of this exercise is to check that the same result is obtained from Maxwell's stress tensor $\overleftrightarrow{T}$, in terms of which the force on a given charge can be written

$$\vec{F} = \oint_{\partial\Omega} \overleftrightarrow{T} \cdot d\vec{a} \qquad \text{(valid for this \underline{static} problem)}$$

where $\partial\Omega$ is the surface enclosing a volume Ω containing only this charge. For symmetry reasons it is convenient to take Ω to be the half-space consisting of all points closer to this charge than to the other charge. Suppose that we want to find the force on the left charge (see figure). The relevant half-space is then the limit $R \to \infty$ of the left hemisphere in the figure.

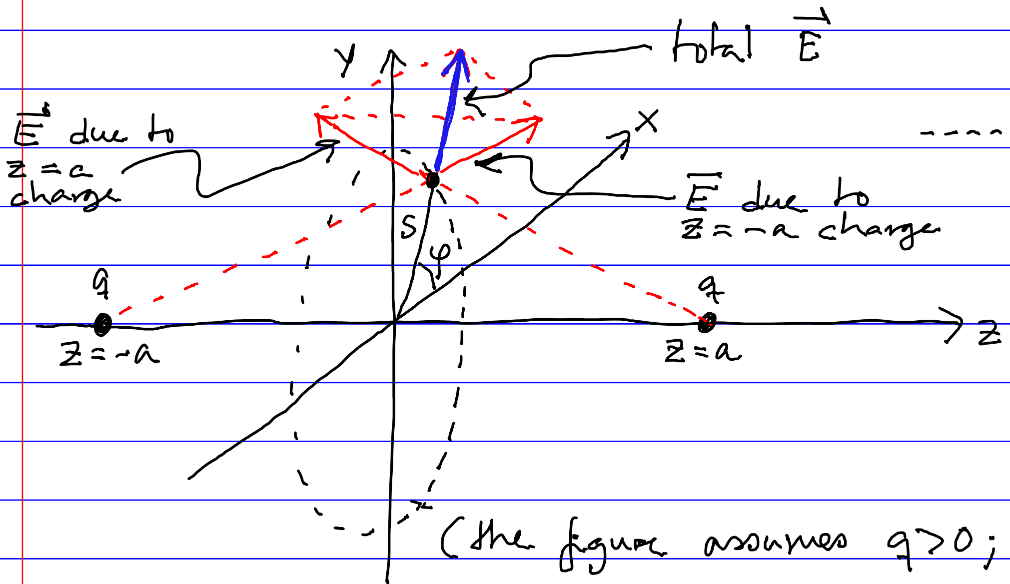


Thus $\partial\Omega$ will consist of two parts: the xy plane and the left half of the spherical surface as $R \to \infty$. Because T_{ij} is quadratic in $\vec{E}$ it will decay as $1/R^4$ (or even faster for cases in which (unlike here) $\vec{E}$ has no monopole term in its multipole expansion), while the area of the spherical surface increases like R^2. Thus the contribution from the spherical surface decays at least as fast as $1/R^2$ and thus vanishes as $R \to \infty$, leaving only the contribution from the xy plane.

If we instead consider the force on the right charge, the spherical surface contribution (now from the right hemisphere) would obviously vanish too. The only difference between the forces on the left charge lies in the direction of $d\vec{a}$ in the xy plane. This should point out of the half space containing the charge. Thus for the left/right charge $d\vec{a}_{L/R} = \pm \hat{z}\, da$ where da is a positive-valued area element. Thus the force components on the left/right charge can be written

$$F_j^{L/R} = \int_{xy\text{ plane}} T_{jk}\, da_k^{L/R} = \pm \int_{xy\text{ plane}} T_{jz}\, da$$

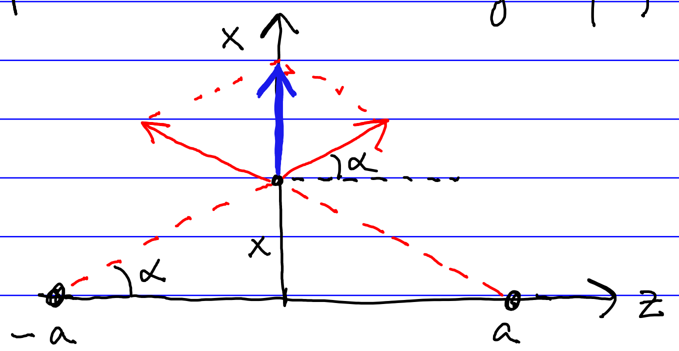
Thus we need to find $\vec{E}$ in the xy plane to construct $\overleftrightarrow{T}$ there. Using cylindrical coordinates (s, φ, z), we can draw the following figure:



- - - - : circle in xy plane, centered in origin, radius s

(the figure assumes $q>0$; reverse all fields if $q<0$)

By symmetry, the total electric field at a point $(r, \varphi, 0)$ in the xy plane will have $E_\varphi = E_z = 0$ while the radial component E_s will be independent of φ. We can therefore calculate it at one particular value of φ, say $\varphi = 0$, when $E_s = E_x$:



$$E_x = \frac{q}{4\pi\epsilon_0(a^2+x^2)} \cdot 2\sin\alpha$$

$$= \frac{q}{2\pi\epsilon_0} \frac{x}{(a^2+x^2)^{3/2}}$$

Thus for a general φ, $E_s = \dfrac{q}{2\pi\epsilon_0} \dfrac{s}{(a^2+s^2)^{3/2}}$

and $E_x = E_s \cos\varphi$, $E_y = E_s \sin\varphi$, $E_z = 0$

This gives $T_{xz} = 0$, $T_{yz} = 0$, and

$$T_{zz} = \epsilon_0\left(E_z^2 - \tfrac{1}{2}E^2\right) = \epsilon_0\left(0 - \tfrac{1}{2}(E_x^2 + E_y^2)\right)$$

$$= -\frac{\epsilon_0}{2} E_s^2 (\cos^2\varphi + \sin^2\varphi) = -\frac{\epsilon_0}{2} E_s^2$$

$\Rightarrow$ $\underline{\underline{F_x^{L/R} = 0}}$, $\underline{\underline{F_y^{L/R} = 0}}$, and

$$\underline{\underline{F_z^{L/R}}} = \pm \int_{\substack{xy\\ \text{plane}}} T_{zz}\, da = \pm \int_0^{2\pi} d\varphi \int_0^\infty ds\, s \left(-\frac{\epsilon_0}{2}\right)\left(\frac{q}{2\pi\epsilon_0}\right)^2 \frac{s^2}{(a^2+s^2)^3}$$

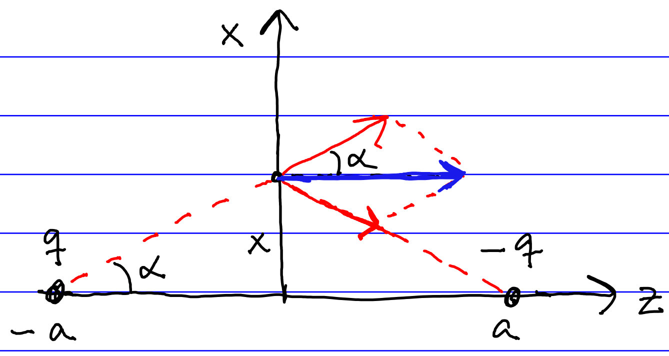
$$= \mp \frac{q^2}{4\pi\epsilon_0} \underbrace{\int_0^\infty ds \frac{s^3}{(a^2+s^2)^3}}_{\frac{1}{4a^2}} = \underline{\underline{\mp \frac{q^2}{4\pi\epsilon_0 (2a)^2}}}$$

which is exactly the expected answer, both in magnitude and sign, expected from Coulomb's law for two like charges q separated by a distance $2a$.

(b) As in (a) we get

$$F_j^{L/R} = \pm \int_{\substack{xy \\ \text{plane}}} T_{jz}\, da$$

But the electric field changes as shown in the figure to the right (drawn for the case $q>0$). We see that for $\varphi = 0$



$$E_x = E_y = 0, \quad E_z = \frac{q}{4\pi\epsilon_0 (a^2+x^2)} \cdot 2\cos\alpha$$

and thus for a general φ,

$$E_x = E_y = 0, \quad E_z = \frac{q}{2\pi\epsilon_0} \frac{a}{(a^2+s^2)^{3/2}}$$

$$\Rightarrow \quad T_{xz} = 0, \quad T_{yz} = 0, \quad \text{and}$$

$$T_{zz} = \epsilon_0 \left(E_z^2 - \frac{1}{2} E^2 \right) = \frac{\epsilon_0}{2} E_z^2$$

Thus $\underline{\underline{F_x^{L/R} = 0}}$, $\underline{\underline{F_y^{L/R} = 0}}$, and

$$\underline{\underline{F_z^{L/R}}} = \pm \frac{\epsilon_0}{2} \left(\frac{q}{2\pi\epsilon_0} \right)^2 a^2 \cdot 2\pi \underbrace{\int_0^\infty ds \frac{s}{(a^2+s^2)^3}}_{1/4a^4} = \underline{\underline{\pm \frac{q^2}{4\pi\epsilon_0 (2a)^2}}}$$

Compared to (a), only the sign has changed, as expected.

Problem 3 (problem 9.11 in Griffiths)

$$f(\vec{r},t) = A\cos(\vec{k}\cdot\vec{r}-\omega t+\delta_a)$$
$$g(\vec{r},t) = B\cos(\underbrace{\vec{k}\cdot\vec{r}-\omega t}_{\equiv\eta}+\delta_b)$$

$$\Rightarrow \langle fg\rangle \equiv \frac{1}{T}\int_0^T dt\, f(\vec{r},t)\, g(\vec{r},t) \qquad \left(T=\frac{2\pi}{\omega}\right)$$

$$= \frac{AB}{T}\int_0^T dt\,[\cos\eta\cos\delta_a - \sin\eta\sin\delta_a][\cos\eta\cos\delta_b - \sin\eta\sin\delta_b]$$

$$= AB\Big[\cos\delta_a\cos\delta_b\underbrace{\langle\cos^2\eta\rangle}_{\frac{1}{2}} + \underbrace{0+0}_{\propto\langle\cos\eta\sin\eta\rangle} + \sin\delta_a\sin\delta_b\cdot\underbrace{\langle\sin^2\eta\rangle}_{\frac{1}{2}}\Big]$$

$$= \frac{AB}{2}(\cos\delta_a\cos\delta_b + \sin\delta_a\sin\delta_b)$$

$$= \underline{\frac{AB}{2}\cos(\delta_a-\delta_b)}$$

On the other hand, defining

$$\tilde{f} = \tilde{A}\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}, \quad \tilde{A} = A\,e^{i\delta_a}$$
$$\tilde{g} = \tilde{B}\,e^{+i(\vec{k}\cdot\vec{r}-\omega t)}, \quad \tilde{B} = B\,e^{i\delta_b}$$

gives $\underline{\frac{1}{2}\mathrm{Re}(\tilde{f}\tilde{g}^*)} = \frac{1}{2}\mathrm{Re}\left(A\,e^{i(\eta+\delta_a)}\,B\,e^{-i(\eta+\delta_b)}\right)$

$$= \frac{AB}{2}\mathrm{Re}\left[e^{i(\delta_a-\delta_b)}\right] = \frac{AB}{2}\cos(\delta_a-\delta_b) = \underline{\langle fg\rangle} \quad \text{QED}$$

If $\delta_a=\delta_b$, $\tilde{f}\tilde{g}^*$ is real, so the "Re" is redundant, giving $\langle fg\rangle = \frac{1}{2}\tilde{f}\tilde{g}^*$ in this case.

Problem 4

(a) $\vec{E}$ and $\vec{B}$ fields of the plane wave:

$$\vec{E}(z,t) = E_0 \cos(kz - \omega t + \delta)\,\hat{x}$$

$$\vec{B}(z,t) = \frac{1}{c} E_0 \cos(kz - \omega t + \delta)\,\hat{y}$$

Maxwell's stress tensor is defined as

$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2\right)$$

For the plane wave only E_x and B_y are nonzero

$\Rightarrow$ all nondiagonal terms of $\overleftrightarrow{T}$ are zero

For the diagonal terms we need

$$\epsilon_0 E^2 = \epsilon_0 E_x^2 = \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)$$

$$\frac{1}{\mu_0} B^2 = \frac{1}{\mu_0} B_y^2 = \frac{1}{\mu_0}\frac{1}{c^2} E_0^2 \cos^2(kz - \omega t + \delta) = \epsilon_0 E^2$$

$$\Rightarrow T_{xx} = \epsilon_0 \left(\underbrace{E_x^2}_{=E^2} - \frac{1}{2}E^2\right) + \frac{1}{\mu_0}\left(\underbrace{B_x^2}_{=0} - \frac{1}{2}B^2\right)$$

$$= \frac{\epsilon_0}{2}E^2 - \frac{\epsilon_0}{2}E^2 = \underline{\underline{0}}$$

$$T_{yy} = \epsilon_0 \left(\underbrace{E_y^2}_{=0} - \frac{1}{2}E^2\right) + \frac{1}{\mu_0}\left(\underbrace{B_y^2}_{=B^2} - \frac{1}{2}B^2\right)$$

$$= -\frac{\epsilon_0}{2}E^2 + \frac{\epsilon_0}{2}E^2 = \underline{\underline{0}}$$

$$T_{zz} = \epsilon_0 \left(\underbrace{E_z^2}_{=0} - \frac{1}{2} E^2 \right) + \frac{1}{\mu_0} \left(\underbrace{B_z^2}_{=0} - \frac{1}{2} B^2 \right)$$

$$= - \frac{\epsilon_0}{2} E^2 \cdot 2 = - \epsilon_0 E^2 = - \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)$$

In conclusion, the only nonzero component of $\overleftrightarrow{T}$ is

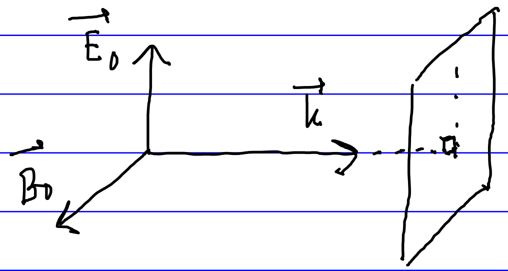
$$T_{zz} = - \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)$$

$-\overleftrightarrow{T}$ is the momentum flux density, which means that $-T_{ij}$ is the momentum in the i direction crossing a surface oriented in the z direction, per unit area, per unit time

Thus for the plane wave,

$$-T_{zz} = \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta) \qquad (= u_{EM})$$

is the z-momentum crossing a surface that is perpendicular to the plane wave, per unit area, per unit time:



(b) The differential form of the momentum conservation is in this case ($\vec{g}_{mech} = 0$ since there are no charges around)

$$\frac{\partial \vec{g}_{EM}}{\partial t} = \nabla \cdot \overleftrightarrow{T} \qquad \left(= -\nabla \cdot \left(-\overleftrightarrow{T}\right)\right)$$

Here $\vec{g}_{EM} = \frac{\vec{S}}{c^2}$. For the plane wave,

$$\vec{S} = c\,u_{EM}\,\hat{z} = c\,\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\,\hat{z}$$

$$\Rightarrow \frac{\partial \vec{g}_{EM}}{\partial t} = \frac{1}{c^2}\frac{\partial \vec{S}}{\partial t} = \frac{1}{c}\hat{z}\frac{\partial u_{EM}}{\partial t}$$

$$= \hat{z}\frac{1}{c}\epsilon_0 E_0^2 \cdot 2\cos(kz-\omega t+\delta)\left(-\sin(kz-\omega t+\delta)\right)\cdot(-\omega)$$

$$= \hat{z}\,k\,\epsilon_0 E^2 \sin[2(kz-\omega t+\delta)]$$

Furthermore,

$$\nabla \cdot \overleftrightarrow{T} = \left(\hat{x}\,\partial_x + \hat{y}\,\partial_y + \hat{z}\,\partial_z\right)\cdot T_{zz}\,\hat{z}\hat{z} = \frac{\partial T_{zz}}{\partial z}\,\hat{z}$$

$$= -\hat{z}\,\epsilon_0 E_0^2 \frac{\partial}{\partial z}\cos^2(kz-\omega t+\delta)$$

$$= -\hat{z}\,\epsilon_0 E_0^2\, 2\cos(kz-\omega t+\delta)\cdot\left(-\sin(kz-\omega t+\delta)\right)\cdot k$$

$$= \hat{z}\,k\,\epsilon_0 E_0^2 \sin[2(kz-\omega t+\delta)]$$

which is the same expression as for $\frac{\partial \vec{g}_{EM}}{\partial t}$. QED.

(c) In this case, since there are no charges around, $\vec{P}_{mech} = 0$, so the conservation law for momentum reads, in integral form,

$$\frac{d}{dt}\int_{\Omega} \vec{g}_{EM}\, d\tau = -\oint_{\partial\Omega} \left(-\overleftrightarrow{T}\right)\cdot d\vec{a}$$

Taking Ω to be a box with lengths L_x, L_y, L_z, the momentum stored in the EM field inside Ω is

$$\vec{P}_{EM} = \int_{\Omega} \vec{g}_{EM}\, d\tau = \underbrace{L_x L_y}_{\equiv A} \int_{z_0}^{z_0+L_z} dz \; \frac{\epsilon_0}{c} E_0^2 \cos^2(kz-\omega t+\delta)\,\hat{z}$$

$$= A \frac{\epsilon_0}{c} E_0^2\, \hat{z} \cdot \left. \frac{2(kz-\omega t+\delta) + \sin[2(kz-\omega t+\delta)]}{4k} \right|_{z_0}^{z_0+L_z}$$

$$= \hat{z} A \frac{\epsilon_0}{c} E_0^2 \frac{1}{4k} \left\{ 2kL_z + \sin[2(k(z_0+L_z)-\omega t+\delta)] - \sin[2(kz_0-\omega t+\delta)] \right\}$$

The time rate of change of this momentum is

$$\frac{d}{dt}\vec{P}_{EM} = -2\omega A \hat{z} \frac{\epsilon_0}{c} E_0^2 \frac{1}{4k} \left\{ \cos[2(k(z_0+L_z)-\omega t+\delta)] - \cos[2(kz_0-\omega t+\delta)] \right\}$$

$\omega = ck$

$$= -\hat{z} \frac{A\epsilon_0 E_0^2}{2} \left\{ \cos[2(k(z_0+L_z)-\omega t+\delta] - \cos[2(kz_0-\omega t+\delta)] \right\}$$

This should be equal to minus $\oint_{\partial\Omega} (-\overleftrightarrow{T})\cdot d\vec{a}$

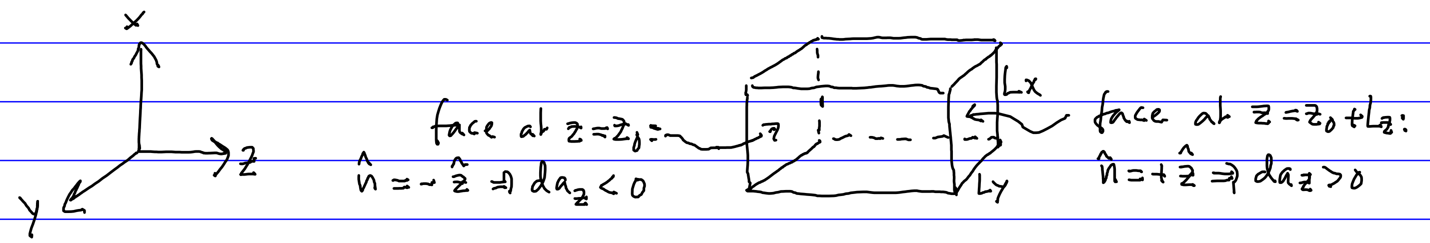
where $\oint_{\partial\Omega} (-\overleftrightarrow{T})\cdot d\vec{a}$ is the momentum flowing out of Ω per unit time. Let us check this:

$$\overleftrightarrow{T} = T_{zz}\,\hat{z}\hat{z} \quad \text{(all other terms of } \overleftrightarrow{T} \text{ are zero)}$$

$$\Rightarrow \oint_{\partial\Omega} \overleftrightarrow{T} \cdot d\vec{a} = \oint_{\partial\Omega} T_{zz}\, \hat{z}\,\underbrace{\hat{z}\cdot\left(da_x\,\hat{x} + da_y\,\hat{y} + da_z\,\hat{z}\right)}_{= \hat{z}\left(da_x \underbrace{\hat{z}\cdot\hat{x}}_{=0} + da_y \underbrace{\hat{z}\cdot\hat{y}}_{=0} + da_z \underbrace{\hat{z}\cdot\hat{z}}_{=1}\right)}$$

$$= \hat{z} \oint_{\partial\Omega} T_{zz}\, da_z$$

The surface integral is a sum of contributions from the 6 faces of $\partial\Omega$. The 4 faces with $\hat{n} = \pm\hat{x}$ or $\hat{n} = \pm\hat{y}$ thus have $da_z = 0$ and therefore contribute zero. The face at $z = z_0$ has $\hat{n} = -\hat{z}$, so $da_z < 0 \Rightarrow da_z = -da$ while the face at $z = z_0 + L_z$ has $\hat{n} = +\hat{z}$, so $da_z > 0$ $\Rightarrow da_z = +da$, where da is a positive-valued area element.



The surface integral therefore becomes

$$\oint_{\partial\Omega} T_{zz}\, da_z = +\int_{z=z_0+L_z} T_{zz}\, da - \int_{z=z_0} T_{zz}\, da$$

Since T_{zz} does not depend on x or y, it is constant over each of the 2 faces and can be taken outside the integral. Using $\int da = L_x L_y = A$ then gives

$$\oint_{\partial\Omega} T_{zz}\, da_z = A\left[T_{zz}(z = z_0 + L_z) - T_{zz}(z = z_0)\right]$$

Inserting for T_{zz} gives $\oint_{\partial\Omega} \overleftrightarrow{T} \cdot d\vec{a}$

$$= -\hat{z} A \epsilon_0 E_0^2 \left[\cos^2\left(k(z_0 + L_z) - \omega t + \delta\right) - \cos^2\left(k z_0 - \omega t + \delta\right)\right]$$

which equals the expression for $d\vec{p}_{EM}/dt$ (seen by using $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$).