TFY4240
Solution problem set 8

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Problem 1.

a) The flux is given by

$$\phi(x) = \int \boldsymbol{B} \cdot d\boldsymbol{a} = \ell \int_0^x ds B_n = \ell x B \sin(\pi/4) = \frac{1}{\sqrt{2}} B \ell x. \tag{1}$$

The magnitude of the induced emf is

$$|\varepsilon| = \left|\frac{d\phi}{dt}\right| = \left|\frac{d\phi}{dx}\frac{dx}{dt}\right| = \frac{1}{\sqrt{2}} B\ell v. \tag{2}$$

The current is therefore

$$I = \frac{|\varepsilon|}{R} = \frac{B\ell v}{\sqrt{2}R}. \tag{3}$$

The current flows clockwise (when the circuit is viewed from above). This can be seen from the magnetic Lorentz force on positive charges in the rod moving with velocity $\boldsymbol{v}$: $\boldsymbol{F}_{\mathrm{q}} = q\boldsymbol{v} \times \boldsymbol{B}$, which points in the $+z$ direction. Alternatively, it can be deduced from Lenz's law: The flux through the loop is increasing, so the induced current will try to oppose this by setting up a magnetic field that points downwards inside the circuit, which (using the right-hand rule for your thumb and curled fingers) implies that the current flows clockwise.

b) The force on the rod is given by

$$\boldsymbol{F}_{\mathrm{rod}} = I\boldsymbol{\ell} \times \boldsymbol{B}, \tag{4}$$

where $\boldsymbol{\ell}$ points in the $+z$ direction (the direction of the current through the rod). The angle between $\boldsymbol{\ell}$ and $\boldsymbol{B}$ is 90°, so the magnitude of the force is

$$|\boldsymbol{F}_{\mathrm{rod}}| = I\ell B = \frac{B^2\ell^2 v}{\sqrt{2}R}. \tag{5}$$

From the right-hand rule, the direction of $\boldsymbol{F}_{\mathrm{rod}}$ is in the xy plane, at a 135° angle with the x axis. Thus the force components are

$$F_{\mathrm{rod,y}} = -F_{\mathrm{rod,x}} = \frac{B^2\ell^2 v}{2R}. \tag{6}$$

As the rod should move in the x direction with constant velocity v, a force of magnitude $|F_{\mathrm{rod,x}}|$ pointing in the positive x direction needs to be applied to the rod by an outside agent in order to balance the negative x component of $\boldsymbol{F}_{\text{rod}}$ (Newton's 1st law). The mechanical power provided by the outside agent is therefore

$$P_{\text{mech}} = |F_{\text{rod,x}}|v = \frac{B^2\ell^2v^2}{2R}. \tag{7}$$

This power is dissipated in the resistor, as the ohmic power loss is

$$P_{\text{electric}} = RI^2 = \frac{B^2\ell^2v^2}{2R} = P_{\text{mech}}. \tag{8}$$

c) The vertical component of the force is given by $F_{\text{rod,y}}$ in (6). The rod levitates when this component is larger than the gravitational force mg, i.e. when

$$B > \sqrt{\frac{2mgR}{\ell^2 v}}. \tag{9}$$

d) Griffiths discusses this problem in Ex. 5.8. (Note that he takes the surface current to flow over the xy plane in the x direction.) Translated to our geometry, Griffiths' conclusion implies that the magnetic field above the conducting plane will point in the negative x direction and have a constant magnitude $B = \mu_0 K/2$.

As an alternative method, we will calculate the magnetic field by considering the uniform surface current to make up a continuous set of straight, parallel, infinitely long wires. We wish to calculate the magnetic field $\boldsymbol{B}$ at an arbitrary point with coordinates (x, y, z). From the symmetry of the problem, the field cannot depend on x or z, so for simplicity we will calculate it for $x = 0$, $z = 0$. Since the magnetic field due to each wire is circumferential, it lies in the xy plane. Thus $B_z = 0$. Furthermore, by adding the contributions from pairs of wires with opposite x coordinates, one can see that the y components of these contributions have opposite sign, so $B_y = 0$. Thus only B_x needs to be explicitly calculated. The current dI through an infinitesimal line segment of length dx in the x direction (i.e. perpendicular to the current direction) is

$$dI = K dx. \tag{10}$$

Thus the contribution to B_x from a wire with x-coordinate x is (see the figure on the next page; the minus sign is because dB_x is seen to be negative)

$$dB_x = -|B_{\text{wire}}| \sin\alpha = \frac{\mu_0 dI}{2\pi a} \sin\alpha \tag{11}$$

where $\sin\alpha = y/a$ and $a^2 = x^2 + y^2$. Thus

$$B_x = \int dB_x = -\frac{\mu_0 K y}{2\pi} \underbrace{\int_{-\infty}^{\infty} \frac{dx}{x^2 + y^2}}_{\pi/|y|} = -\frac{\mu_0 K}{2}\,\text{sgn}(y), \tag{12}$$

where $\text{sgn}(y) \equiv y/|y|$ is the sign of y. This gives the same result for the field above the zx plane (i.e. $y > 0$) as noted earlier.

e) When the direction of the current I is opposite to that of $\boldsymbol{K}$, the force $I\boldsymbol{\ell} \times \boldsymbol{B}$ will be upwards. Levitation requires $I\ell B > mg$, i.e.

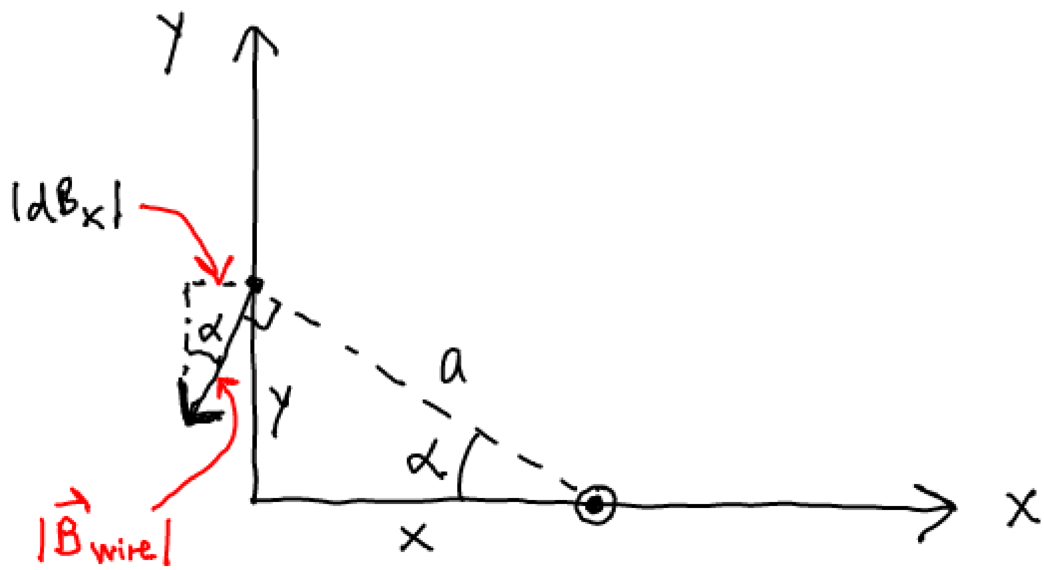
$$I > \frac{mg}{B\ell} = \frac{9.8\text{m/s}^2}{2.0\text{ T}} \cdot \frac{1000\text{ kg}}{1\text{ m}} = 4.9 \cdot 10^3\text{ A}. \tag{13}$$

Problem 2.
See handwritten solution further back.

Problem 3.
See Griffiths.

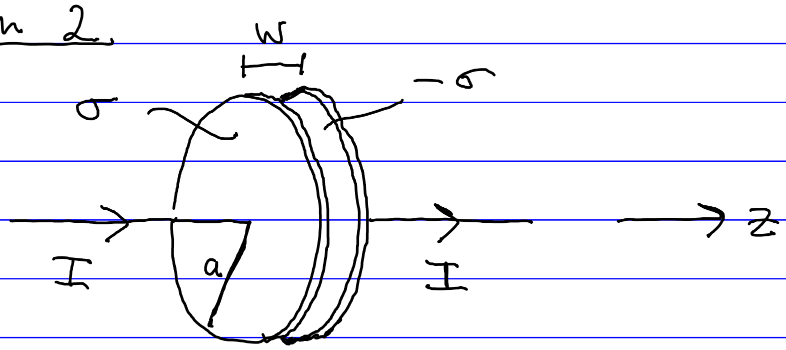
Problem 4.
See handwritten solution further back.

Problem 5.
See handwritten solution further back.



Problem 2.

(a)



The electric field in the gap is normal to the plates, pointing from the positively to the negatively charged plate, and has magnitude

$$E = \frac{\sigma}{\epsilon_0}$$

where $\pm\sigma$ is the surface charge density the two plates (see Ex. 2.5 and Ex. 2.10 in Griffiths; this expression neglects the "fringing field" around the edges; see Sec. 4.4.4).

The charge $\pm Q$ on the plates is given by $Q = \sigma A$ with $A = \pi a^2$. With $Q = 0$ at time $t = 0$ and a time-independent current I we get $Q = It$
$\Rightarrow \sigma = It/A$

$$\Rightarrow \quad \underline{\underline{E = \frac{It}{\epsilon_0 A}}}$$

To find $\vec{B}$ we consider a circular Amperian loop C' of radius s inside the gap, oriented $\parallel$ to the plates and with a center on the axis passing through both plate centers. We take the circular disk of radius s bounded by C' as the surface S' in Stokes's theorem.

Then the Ampere-Maxwell law gives

$$\oint_{C'} \vec{B} \cdot d\vec{l} = \mu_0 \int_{S'} \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{a}$$

Since there is no current in the gap, $\vec{J} = 0$ on S'.
Furthermore,

$$\frac{\partial \vec{E}}{\partial t} = \frac{I}{\epsilon_0 A} \quad \text{(a constant)}$$

$$\Rightarrow \mu_0 \epsilon_0 \int \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a} = \mu_0 \cancel{\epsilon_0} \frac{I}{\cancel{\epsilon_0} A} \cdot \pi s^2 = \mu_0 I \left(\frac{s}{a} \right)^2$$

By symmetry, $\vec{B}$ will be tangential on C' and have a magnitude that only depends on s. Thus

$$\oint_{C'} \vec{B} \cdot d\vec{l} = B \cdot 2\pi s$$

$$\Rightarrow \quad B = \frac{\mu_0 I}{2\pi s} \left(\frac{s}{a} \right)^2 = \underline{\underline{\frac{\mu_0 I s}{2A}}}$$

The direction of $\vec{B}$ is clockwise around C' when viewed from the positive plate (this follows from a right-hand rule relating the positive circulation direction around C' to the positive direction of S').

(I called the surface S' to avoid confusion with the Poynting vector $\vec{S}$ and its magnitude S.)

(b) The energy density u_{EM} in the gap is

$$u_{EM} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

$$= \frac{1}{2}\left(\epsilon_0 \left(\frac{It}{\epsilon_0 A}\right)^2 + \frac{1}{\mu_0}\left(\frac{\mu_0 I s}{2A}\right)^2\right)$$

$$= \underline{\underline{\frac{I^2}{2A^2}\left(\frac{t^2}{\epsilon_0} + \frac{\mu_0 s^2}{4}\right)}}$$

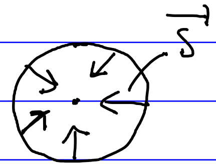
The Poynting vector $\vec{S}$ in the gap is

$$\vec{S} = \frac{1}{\mu_0}\left(\vec{E} \times \vec{B}\right)$$

Introducing cylindrical coordinates (s, ϕ, z) with the z axis going through the plate centers, we find that $\vec{S}$ points in the negative radial direction (ie towards the z axis) and has magnitude

$$S = \frac{1}{\mu_0} EB \sin\frac{\pi}{2} = \frac{1}{\cancel{\mu_0}} \frac{It}{\epsilon_0 A} \frac{\cancel{\mu_0} I s}{2A} = \frac{I^2 s t}{2\epsilon_0 A^2}$$

i.e. $\underline{\underline{\vec{S} = -\frac{I^2 s t}{2\epsilon_0 A^2}\,\hat{e}_s}} \quad \left(\equiv S_s \hat{e}_s\right)$



(c) Since there are no charges in the gap, $u_{mech} = 0 \Rightarrow \frac{\partial u_{mech}}{\partial t} = 0$. From (b) we find

$$\frac{\partial u_{EM}}{\partial t} = \underline{\underline{\frac{I^2 t}{A^2 \epsilon_0}}}$$

looking up expression for the divergence in cylindrical coordinates

and $-\nabla\cdot\vec{S} = -\frac{1}{s}\frac{\partial}{\partial s}\left(s S_s\right) = \frac{I^2 t}{2\epsilon_0 A^2}\frac{1}{s}\frac{\partial}{\partial s}\left(s^2\right) = \underline{\underline{\frac{I^2 t}{\epsilon_0 A^2}}} = \frac{\partial u_{EM}}{\partial t}$

Q.E.D.

(d) Since there are no charges in the gap, $\frac{dW}{dt} = 0$.
Furthermore,

$$\frac{d}{dt}\int_\Omega u_{EM}\, d\tau = \frac{d}{dt}\int_\Omega \frac{I^2}{2A^2}\left(\frac{t^2}{\epsilon_0} + \underbrace{\frac{\mu_0 s^2}{4}}_{\text{time-indep } \Rightarrow \frac{d}{dt} \to 0}\right) d\tau$$

$$= \frac{I^2}{2A^2}\frac{1}{\epsilon_0}\cdot 2t \cdot w \cdot \pi b^2 = \underline{\underline{\frac{I^2 \pi b^2 w t}{\epsilon_0 A^2}}}$$

Integrating $\vec{S}$ over the boundary $\partial\Omega$ of Ω, there is no contribution from the two end faces since $\vec{S}\cdot d\vec{a} = 0$ there. Thus the only contribution comes from the curved part where $\hat{n}$ points in the opposite direction of $\vec{S}$, and where S is constant $(s=b)$

$$\Rightarrow \; -\oint_{\partial\Omega} \vec{S}\cdot d\vec{a} = -(-1)\frac{I^2 b t}{2\epsilon_0 A^2}\cdot w \cdot 2\pi b = \underline{\underline{\frac{I^2 \pi b^2 w t}{\epsilon_0 A^2}}}$$

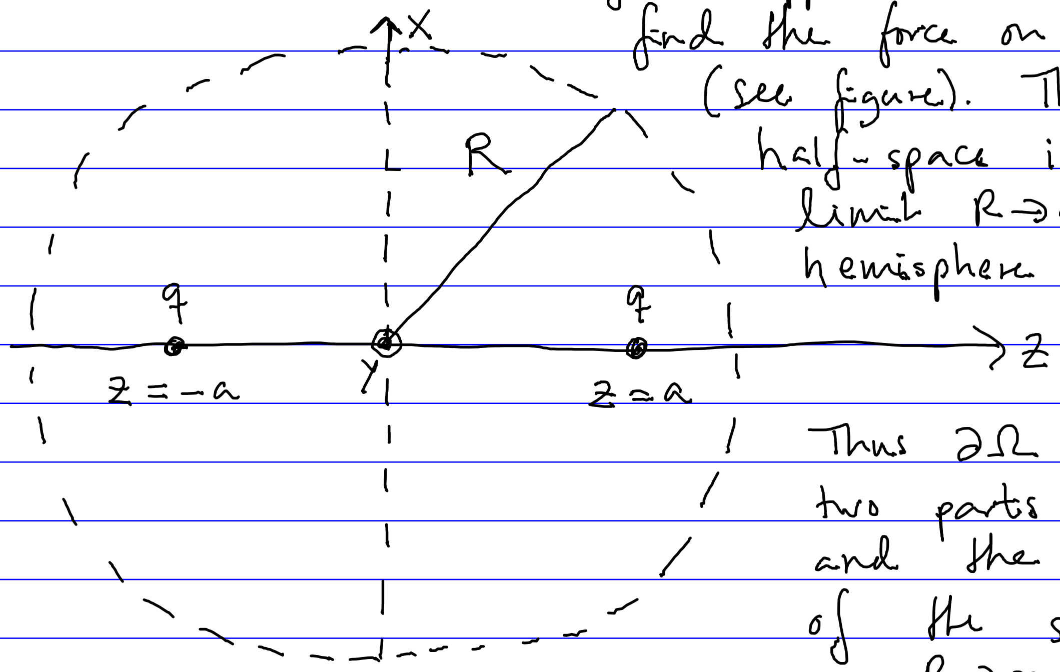
$$= \frac{d}{dt}\int_\Omega u_{EM}\, d\tau \qquad \text{Q.E.D.}$$

Problem 4 (problem 8.4 in Griffiths)

(a) Of course, in this problem the force is already known; it is just given by Coulomb's law. But the point of this exercise is to check that the same result is obtained from Maxwell's stress tensor $\overleftrightarrow{T}$, in terms of which the force on a given charge can be written

$$\vec{F} = \oint_{\partial\Omega} \overleftrightarrow{T} \cdot d\vec{a} \qquad \text{(valid for this \underline{static} problem)}$$

where $\partial\Omega$ is the surface enclosing a volume Ω containing only this charge. For symmetry reasons it is convenient to take Ω to be the half-space consisting of all points closer to this charge than to the other charge. Suppose that we want to find the force on the left charge (see figure). The relevant half-space is then the limit $R \to \infty$ of the left hemisphere in the figure.

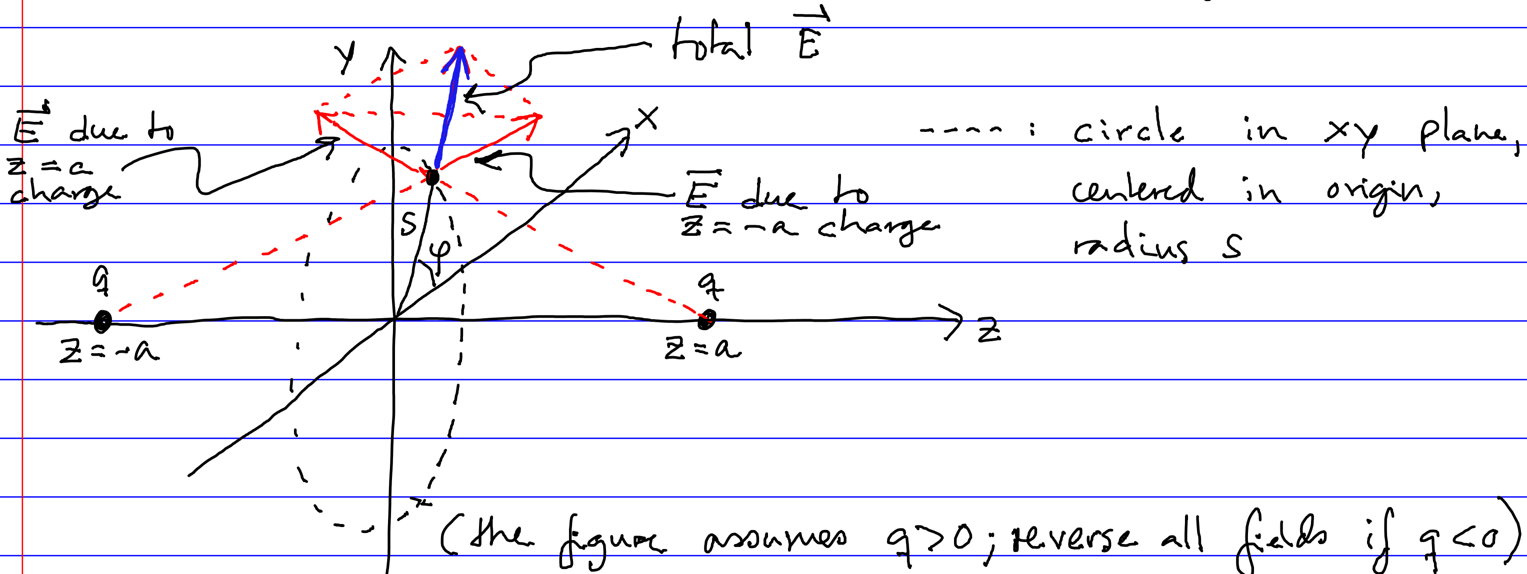


Thus $\partial\Omega$ will consist of two parts: the xy plane and the left half of the spherical surface as $R \to \infty$. Because T_{ij} is quadratic in $\vec{E}$ it will decay as $1/R^4$ (or even faster for cases in which (unlike here) $\vec{E}$ has no monopole term in its multipole expansion), while the area of the spherical surface increases like R^2. Thus the contribution from the spherical surface decays at least as fast as $1/R^2$ and thus vanishes as $R \to \infty$, leaving only the contribution from the xy plane.

If we instead consider the force on the right charge, the spherical surface contribution (now from the right hemisphere) would obviously vanish too. The only difference between the forces on the left charge lies in the direction of $d\vec{a}$ in the xy plane. This should point out of the half space containing the charge. Thus for the left/right charge $d\vec{a}_{L/R} = \pm \hat{z}\, da$ where da is a positive-valued area element. Thus the force components on the left/right charge can be written

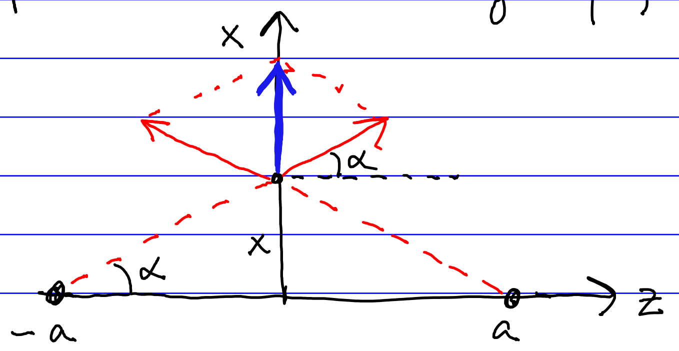
$$F_j^{L/R} = \int_{xy\ \text{plane}} T_{jk}\, da_k^{L/R} = \pm \int_{xy\ \text{plane}} T_{jz}\, da$$

Thus we need to find $\vec{E}$ in the xy plane to construct $\overleftrightarrow{T}$ there. Using cylindrical coordinates (s, φ, z), we can draw the following figure:



(the figure assumes $q>0$; reverse all fields if $q<0$)

By symmetry, the total electric field at a point $(r, \varphi, 0)$ in the xy plane will have $E_\varphi = E_z = 0$ while the radial component E_s will be independent of φ. We can therefore calculate it at one particular value of φ, say $\varphi = 0$, when $E_s = E_x$:



$$E_x = \frac{q}{4\pi\epsilon_0(a^2+x^2)} \cdot 2\sin\alpha$$

$$= \frac{q}{2\pi\epsilon_0} \frac{x}{(a^2+x^2)^{3/2}}$$

Thus for a general φ, $E_s = \dfrac{q}{2\pi\epsilon_0} \dfrac{s}{(a^2+s^2)^{3/2}}$

and $E_x = E_s \cos\varphi$, $E_y = E_s \sin\varphi$, $E_z = 0$

This gives $T_{xz} = 0$, $T_{yz} = 0$, and

$$T_{zz} = \epsilon_0\left(E_z^2 - \tfrac{1}{2}E^2\right) = \epsilon_0\left(0 - \tfrac{1}{2}(E_x^2 + E_y^2)\right)$$

$$= -\frac{\epsilon_0}{2} E_s^2 (\cos^2\varphi + \sin^2\varphi) = -\frac{\epsilon_0}{2} E_s^2$$

$\Rightarrow$ $\underline{\underline{F_x^{L/R} = 0}}$, $\underline{\underline{F_y^{L/R} = 0}}$, and

$$\underline{\underline{F_z^{L/R}}} = \pm \int_{\substack{xy\\ \text{plane}}} T_{zz}\, da = \pm \int_0^{2\pi} d\varphi \int_0^\infty ds\, s \left(-\frac{\epsilon_0}{2}\right)\left(\frac{q}{2\pi\epsilon_0}\right)^2 \frac{s^2}{(a^2+s^2)^3}$$

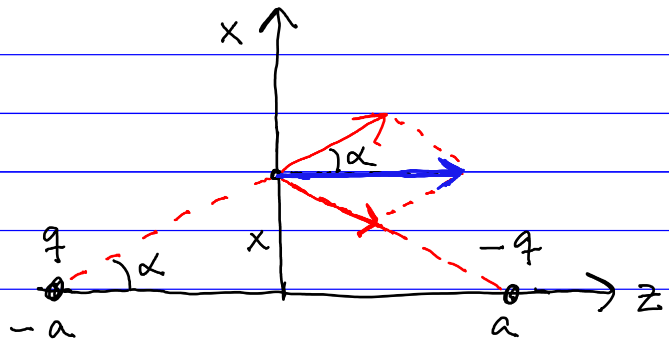
$$= \mp \frac{q^2}{4\pi\epsilon_0} \underbrace{\int_0^\infty ds \frac{s^3}{(a^2+s^2)^3}}_{\frac{1}{4a^2}} = \underline{\underline{\mp \frac{q^2}{4\pi\epsilon_0 (2a)^2}}}$$

which is exactly the expected answer, both in magnitude and sign, expected from Coulomb's law for two like charges q separated by a distance $2a$.

(b) As in (a) we get

$$F_j^{L/R} = \pm \int_{\substack{xy \\ \text{plane}}} T_{jz}\, da$$

But the electric field changes as shown in the figure to the right (drawn for the case $q>0$). We see that for $\varphi = 0$



$$E_x = E_y = 0 \,, \quad E_z = \frac{q}{4\pi\epsilon_0 (a^2+x^2)} \cdot 2\cos\alpha$$

and thus for a general φ,

$$E_x = E_y = 0 \,, \quad E_z = \frac{q}{2\pi\epsilon_0} \frac{a}{(a^2+s^2)^{3/2}}$$

$$\Rightarrow \quad T_{xz} = 0 \,, \quad T_{yz} = 0 \,, \quad \text{and}$$

$$T_{zz} = \epsilon_0 \left(E_z^2 - \frac{1}{2} E^2 \right) = \frac{\epsilon_0}{2} E_z^2$$

Thus $\underline{\underline{F_x^{L/R} = 0}}$, $\underline{\underline{F_y^{L/R} = 0}}$, and

$$\underline{\underline{F_z^{L/R}}} = \pm \frac{\epsilon_0}{2} \left(\frac{q}{2\pi\epsilon_0} \right)^2 a^2 \cdot 2\pi \underbrace{\int_0^\infty ds \frac{s}{(a^2+s^2)^3}}_{1/4a^4} = \underline{\underline{\pm \frac{q^2}{4\pi\epsilon_0 (2a)^2}}}$$

Compared to (a), only the sign has changed, as expected.

Problem 5 (problem 9.1) in Griffiths)

$$f(\vec{r},t) = A\cos(\vec{k}\cdot\vec{r} - \omega t + \delta_a)$$
$$g(\vec{r},t) = B\cos(\underbrace{\vec{k}\cdot\vec{r} - \omega t}_{\equiv \eta} + \delta_b)$$

$$\Rightarrow \langle f g\rangle \equiv \frac{1}{T}\int_0^T dt\; f(\vec{r},t)\, g(\vec{r},t) \qquad \left(T = \frac{2\pi}{\omega}\right)$$

$$= \frac{AB}{T}\int_0^T dt\,[\cos\eta\,\cos\delta_a - \sin\eta\,\sin\delta_a][\cos\eta\,\cos\delta_b - \sin\eta\,\sin\delta_b]$$

$$= AB\Big[\cos\delta_a\cos\delta_b\underbrace{\langle\cos^2\eta\rangle}_{\frac{1}{2}} + \underbrace{0 + 0}_{\propto\langle\cos\eta\sin\eta\rangle} + \sin\delta_a\sin\delta_b\cdot\underbrace{\langle\sin^2\eta\rangle}_{\frac{1}{2}}\Big]$$

$$= \frac{AB}{2}\left[\cos\delta_a\cos\delta_b + \sin\delta_a\sin\delta_b\right]$$

$$= \frac{AB}{2}\cos(\delta_a - \delta_b)$$

On the other hand, defining

$$\tilde{f} = \tilde{A}\,e^{i(\vec{k}\cdot\vec{r} - \omega t)}, \qquad \tilde{A} = A\,e^{i\delta_a}$$
$$\tilde{g} = \tilde{B}\,e^{+i(\vec{k}\cdot\vec{r} - \omega t)}, \qquad \tilde{B} = B\,e^{i\delta_b}$$

gives $\frac{1}{2}\mathrm{Re}(\tilde{f}\,\tilde{g}^*) = \frac{1}{2}\mathrm{Re}\left(A\,e^{i(\eta+\delta_a)}\,B\,e^{-i(\eta+\delta_b)}\right)$

$$= \frac{AB}{2}\mathrm{Re}\left[e^{i(\delta_a-\delta_b)}\right] = \frac{AB}{2}\cos(\delta_a - \delta_b) = \langle fg\rangle \quad \text{QED}$$

If $\delta_a = \delta_b$, $\tilde{f}\tilde{g}^*$ is real, so the "Re" is redundant, giving $\langle fg\rangle = \frac{1}{2}\tilde{f}\tilde{g}^*$ in this case.