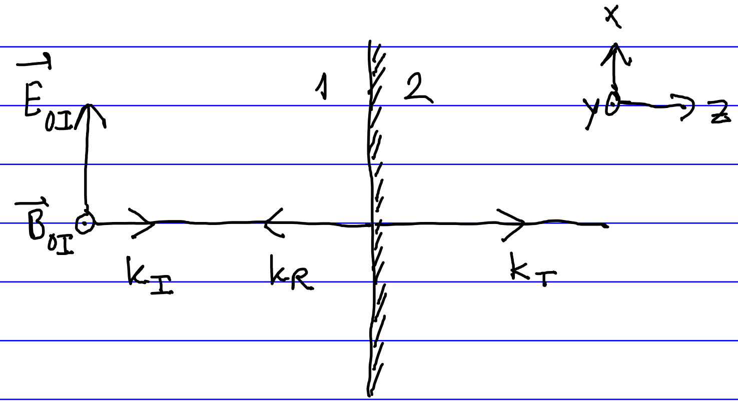
TFY4240 Tutorial 9, solution

Problem 1



Normal incidence assumed, ie. $\theta_I = \theta_R = \theta_T = 0$

Boundary conditions (BC's):
(i) $\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}$
(ii) $B_1^{\perp} = B_2^{\perp}$
(iii) $\vec{E}_1^{\,\parallel} = \vec{E}_2^{\,\parallel}$
(iv) $\vec{B}_1^{\,\parallel}/\mu_1 = \vec{B}_2^{\,\parallel}/\mu_2$

Since all $\vec{k}$-vectors ($\vec{k}_I$, $\vec{k}_R$, and $\vec{k}_T$) are $\perp$ to interface, and since both $\vec{E}$ and $\vec{B}$ are transverse (ie. $\vec{E} \perp \vec{k}$, $\vec{B} \perp \vec{k}$), we have $E_\perp = 0$ and $B_\perp = 0$, and therefore also $\vec{E}^{\parallel} = \vec{E}$ and $\vec{B}^{\parallel} = \vec{B}$ for all waves (incident, reflected and transmitted). Thus BC's (i) and (ii) are trivially satisfied, and BC's (iii) and (iv) become

(iii') $\vec{E}_1 = \vec{E}_2$
(iv') $\vec{B}_1/\mu_1 = \vec{B}_2/\mu_2$

To consider (iii') we use $\vec{E}_1 = \vec{E}_I + \vec{E}_R$, $\vec{E}_2 = \vec{E}_T$, where

Incident $\vec{E}$: $\tilde{\vec{E}}_I(z,t) = \tilde{E}_{0I}\, e^{i(k_1 z - \omega t)}\, \hat{x}$

Reflected $\vec{E}$: $\tilde{\vec{E}}_R(z,t) = \tilde{E}_{0R}\, e^{i(-k_1 z - \omega t)}\, \hat{n}_R$

Transmitted $\vec{E}$: $\tilde{\vec{E}}_T(z,t) = \tilde{E}_{0T}\, e^{i(k_2 z - \omega t)}\, \hat{n}_T$

where $\hat{n}_R = \hat{x}\cos\varphi_R + \hat{y}\sin\varphi_R$

$\hat{n}_T = \hat{x}\cos\varphi_T + \hat{y}\sin\varphi_T$

At the boundary, $z=0 \Rightarrow e^{i(\cdots\cdots)} = e^{-i\omega t}$ is the same for all fields and can be cancelled. Thus (iii') gives

$$\tilde{E}_{0I}\,\hat{x} + \tilde{E}_{0R}\,\hat{n}_R = \tilde{E}_{0T}\,\hat{n}_T$$

$$\Rightarrow \tilde{E}_{0I}\,\hat{x} + \tilde{E}_{0R}\left(\cos\varphi_R\,\hat{x} + \sin\varphi_R\,\hat{y}\right) = \tilde{E}_{0T}\left(\cos\varphi_T\,\hat{x} + \sin\varphi_T\,\hat{y}\right).$$

Separately equating the x & y components of this vector equation gives

$$\tilde{E}_{0I} + \tilde{E}_{0R}\cos\varphi_R = \tilde{E}_{0T}\cos\varphi_T \qquad \text{(iii'-a)}$$

$$\tilde{E}_{0R}\sin\varphi_R = \tilde{E}_{0T}\sin\varphi_T \qquad \text{(iii'-b)}$$

To consider (iv') we use that $\vec{B}_1 = \vec{B}_I + \vec{B}_R$, $\vec{B}_2 = \vec{B}_T$, where

Incident $\vec{B}$: $\tilde{\vec{B}}_I(z,t) = \frac{1}{v_1}\hat{k}_I \times \tilde{\vec{E}}_I(z,t)$

$$= \frac{1}{v_1}\hat{z} \times \tilde{E}_{0I}\, e^{i(k_1 z - \omega t)}\,\hat{x}$$

$$= \frac{1}{v_1}\tilde{E}_{0I}\, e^{i(k_1 z - \omega t)}\,\hat{y}$$

Reflected $\vec{B}$: $\tilde{\vec{B}}_R(z,t) = \frac{1}{v_1}\hat{k}_R \times \tilde{\vec{E}}_R(z,t)$

$$= \frac{1}{v_1}(-\hat{z}) \times \tilde{E}_{0R}\, e^{i(-k_1 z - \omega t)}\left[\hat{x}\cos\varphi_R + \hat{y}\sin\varphi_R\right]$$

$$= \frac{1}{v_1}\tilde{E}_{0R}\, e^{i(-k_1 z - \omega t)}\left[-\hat{y}\cos\varphi_R + \hat{x}\sin\varphi_R\right]$$

Transmitted $\vec{B}$: $\vec{B}_T(z,t) = \frac{1}{v_2}\vec{k}_T \times \tilde{\vec{E}}_T(z,t)$

$$= \frac{1}{v_2}\hat{z} \times \tilde{E}_{0T}\, e^{i(k_2 z - \omega t)}\left[\hat{x}\cos\varphi_T + \hat{y}\sin\varphi_T\right]$$

$$= \frac{1}{\sqrt{2}} \tilde{E}_{0T}\, e^{i(k_2 z - \omega t)} \left[\hat{y} \cos\varphi_T - \hat{x} \sin\varphi_T \right]$$

At the boundary the $e^{i(\dots)}$ factors are again identical and thus cancel in (iv'), which gives

$$\frac{1}{\mu_1 v_1} \left[\hat{x}\, \tilde{E}_{0R} \sin\varphi_R + \hat{y} \left(\tilde{E}_{0I} - \tilde{E}_{0R} \cos\varphi_R \right) \right]$$
$$= \frac{1}{\mu_2 v_2} \left[-\hat{x}\, \tilde{E}_{0T} \sin\varphi_T + \hat{y}\, \tilde{E}_{0T} \cos\varphi_T \right]$$

which gives the following equations for the x and y components

$$\frac{1}{\mu_1 v_1} \tilde{E}_{0R} \sin\varphi_R = -\frac{1}{\mu_2 v_2} \tilde{E}_{0T} \sin\varphi_T \qquad \text{(iv'-a)}$$
$$\frac{1}{\mu_1 v_1} \left(\tilde{E}_{0I} - \tilde{E}_{0R} \cos\varphi_R \right) = \frac{1}{\mu_2 v_2} \tilde{E}_{0T} \cos\varphi_T \qquad \text{(iv'-b)}$$

If $\sin\varphi_R \neq 0$, (iii'-b) and (iv'-a) imply $\sin\varphi_T \neq 0$ as well. Assuming $\sin\varphi_R, \sin\varphi_T \neq 0$, divide (iii'-b) by (iv'-a). This gives $\mu_1 v_1 = -\mu_2 v_2$, which is an equation without physical solutions (at least for conventional materials). If we nevertheless accept it for now, insert it into (iii'-a) and (iv'-b), and divide the former by the latter, we get (with $Z \equiv (\tilde{E}_{0R}/\tilde{E}_{0I}) \cos\varphi_R$)

$$(1+Z)/(1-Z) = -1 \Rightarrow 1 + Z = Z - 1 \Rightarrow 1 = -1$$

which obviously isn't even mathematically possible. Thus we conclude from (iii'-b) and (iv'-a) that $\sin\varphi_R = \sin\varphi_T = 0$ $\Rightarrow \cos\varphi_R, \cos\varphi_T = \pm 1$, corresponding to angles 0 or π, which gives $\hat{n}_R, \hat{n}_T = \pm\hat{x}$. But the <u>sign</u> of $\hat{n}$ is not physical (see footnote 2 in Ch. 9 of Griffiths; another way of seeing this is that we have just 2 eqs left [(iii'-a) & (iv'-b)], so there cannot be more unknowns than $\tilde{E}_{0R}$ and $\tilde{E}_{0T}$. So we just pick the signs. The most convenient choice is $\hat{n}_R = \hat{n}_T = +\hat{x}$, i.e. $\varphi_R = \varphi_T = 0$.

Problem 2

We have

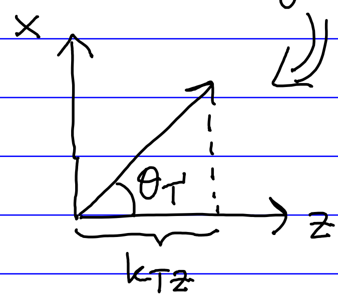
$$\tilde{\vec{E}}_T = \tilde{\vec{E}}_{0T}\, e^{i(\vec{k}_T\cdot\vec{r}-\omega t)}$$

where $\vec{k}_T = (k_{Tx}, k_{Ty}, k_{Tz})$

$$\begin{aligned} k_{Tx} &= k_{Ix} \quad \text{(Griffiths Eq. (9.97))} \\ &= k_I \sin\theta_I \\ &= (\omega/v_1)\sin\theta_I = \frac{\omega}{c} n_1 \sin\theta_I \equiv k \quad \text{(real)} \end{aligned}$$

$$\begin{aligned} k_{Ty} &= k_{Iy} \quad \text{(Griffiths Eq. (9.96))} \\ &= 0 \end{aligned}$$

$k_{Tz} = k_T \cos\theta_T$ ← from the geometric def of θ_T



$$\Rightarrow k_{Tz}^2 = k_T^2 \cos^2\theta_T$$

$$= \left(\frac{\omega}{c} n_2\right)^2 (1-\sin^2\theta_T)$$

$$= \left(\frac{\omega}{c}\right)^2 n_2^2 \left(1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_I\right)$$

$$= \left(\frac{\omega}{c}\right)^2 \left(n_2^2 - n_1^2 \sin^2\theta_I\right)$$

$$= \left(\frac{\omega}{c}\right)^2 n_1^2 \left[\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_I\right]$$

$$= \left(\frac{\omega}{c}\right)^2 n_1^2 \left[\sin^2\theta_c - \sin^2\theta_I\right]$$

If $\theta_I > \theta_c$, we see that $k_{Tz}^2 < 0$, so k_{Tz} is imaginary $\Rightarrow k_{Tz} = \pm i\kappa$ with

$$\kappa \equiv \frac{\omega}{c} n_1 \sqrt{\sin^2\theta_I - \sin^2\theta_c} \quad \text{(real and positive)}$$

Inserting this into $e^{i\vec{k}_T \cdot \vec{r}}$, the z-dependence becomes

$$e^{i(\pm i\kappa)z} = e^{\mp \kappa z}$$

The bottom sign implies exponential growth of the transmitted wave as it propagates further into medium 2. This does not make sense physically, so the top sign must be chosen, i.e. $k_{Tz} = +i\kappa$. Therefore

$$\tilde{\vec{E}}_T = \tilde{\vec{E}}_{0T}\, e^{-\kappa z}\, e^{i(kx - \omega t)} \qquad \text{(in medium 2, ie } z \geq 0)$$

(we note that $k_{Tz} = k_T \cos\theta_T = + i\kappa$ gives $\cos\theta_T = i\frac{\kappa}{k_T}$ which is a number on the positive imaginary axis, giving $\cos\theta_T = + i\sqrt{\sin^2\theta_T - 1}$ as stated in the problem text).

(b) We have, for p-polarization

$$\frac{\tilde{E}_{0R}}{\tilde{E}_{0I}} = \frac{\alpha - \beta}{\alpha + \beta} = \frac{\frac{\cos\theta_T}{\cos\theta_I} - \beta}{\frac{\cos\theta_T}{\cos\theta_I} + \beta} = \frac{\cos\theta_T - \beta\cos\theta_I}{\cos\theta_T + \beta\cos\theta_I}$$

$$= \frac{ic - \beta\cos\theta_I}{ic + \beta\cos\theta_I} \qquad \left(\text{where } c \equiv \frac{\kappa}{k_T} \text{ is real}\right)$$

$$= \frac{c + i\beta\cos\theta_I}{c - i\beta\cos\theta_I} \qquad (\text{note that } \beta\cos\theta_I \text{ is real too})$$

Define the complex number $w = c + i\beta\cos\theta_I \equiv \rho e^{i\phi}$ where $\rho = |w|$ and $\phi = \arg(w)$

$$\Rightarrow \frac{\tilde{E}_{0R}}{\tilde{E}_{0I}} = \frac{\rho e^{i\phi}}{\rho e^{-i\phi}} = e^{2i\phi},$$ i.e. a complex number of magnitude 1 ("unit modulus")

The reflection coefficient R is

$$R \equiv \frac{I_R}{I_I} = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \frac{|\tilde{E}_{0R}|^2}{|E_{0I}|^2} = \left|\frac{\tilde{E}_{0R}}{\tilde{E}_{0I}}\right|^2 \qquad (*)$$

where E_{0R} and E_{0I} are the (real) amplitudes of the reflected and incident waves, and we used $E_{0R} = |\tilde{E}_{0R}|$ and $E_{0I} = |\tilde{E}_{0I}|$. (Note that $(*)$ corresponds to

$R = \left|\frac{\alpha-\beta}{\alpha+\beta}\right|^2$, which generalizes Eq. (9.115) in Griffiths

to situations with α complex.) For our case we thus get

$$\underline{\underline{R}} = |e^{2i\phi}|^2 = \underline{\underline{1}}$$

(c) The time-averaged Poynting vector for the transmitted wave is

$$\langle \vec{S}_T \rangle = \frac{1}{2\mu_2} \mathrm{Re}\left[\tilde{\vec{E}}_T^* \times \tilde{\vec{B}}_T\right]$$

Using $\tilde{\vec{E}}_T = \tilde{\vec{E}}_{0T}\, e^{i(\vec{k}_T\cdot\vec{r}-\omega t)}$ and Faraday's law,

$$\nabla \times \tilde{\vec{E}}_T = -\tilde{\vec{E}}_{0T} \times \nabla e^{i(\vec{k}_T\cdot\vec{r}-\omega t)} \qquad \text{(product rule 7)}$$

$$= i\vec{k}_T \times \tilde{\vec{E}}_T = -(-i\omega)\tilde{\vec{B}}_T$$

$$\Rightarrow \tilde{\vec{B}}_T = \frac{1}{\omega} \vec{k}_T \times \tilde{\vec{E}}_T$$

$$\Rightarrow \langle \vec{S}_T \rangle = \frac{1}{2\mu_2\omega} \mathrm{Re}\left[\tilde{\vec{E}}_T^* \times (\vec{k}_T \times \tilde{\vec{E}}_T)\right]$$

Using the triple product rule, this becomes

$$\langle \vec{S}_T \rangle = \frac{1}{2\mu_2 \omega} \operatorname{Re}\left[\vec{k}_T (\tilde{\vec{E}}_T \cdot \tilde{\vec{E}}_T) - \tilde{\vec{E}}_T (\tilde{\vec{E}}_T^* \cdot \vec{k}_T) \right]$$

where $\tilde{\vec{E}}_T^* \cdot \tilde{\vec{E}}_T = |\tilde{\vec{E}}_T|^2$ (real) and

$$\tilde{\vec{E}}_T^* \cdot \vec{k}_T = \tilde{E}_{Tx}^* k_{Tx} + \tilde{E}_{Tz}^* k_{Tz} \quad \text{(y-components are 0)}$$

$$= \tilde{E}_{Tx}^* k_{Tx} + \tilde{E}_{Tz}^* i\kappa$$

From $\nabla \cdot \tilde{\vec{E}}_T = 0$ follows $\vec{k}_T \cdot \tilde{\vec{E}}_T = 0$.
Taking the complex conjugate gives

$$0 = \vec{k}_T^* \cdot \tilde{\vec{E}}_T^* = \tilde{E}_{Tx}^* k_{Tx} - \tilde{E}_{Tz}^* i\kappa = 0$$

$$\Rightarrow \tilde{\vec{E}}_T^* \cdot \vec{k}_T = 2 i \kappa \tilde{E}_{Tz}^*$$

$$\Rightarrow \langle \vec{S}_T \rangle = \frac{1}{2\mu_2 \omega} \operatorname{Re}\left[\vec{k}_T |\tilde{\vec{E}}_T|^2 - \tilde{\vec{E}}_T \, 2i\kappa \tilde{E}_{Tz}^* \right]$$

$$\Rightarrow \langle \vec{S}_T \rangle \cdot \hat{z} = \frac{1}{2\mu_2 \omega} \operatorname{Re}\left[k_{Tz} |\tilde{\vec{E}}_T|^2 - \tilde{E}_{Tz} \, 2i\kappa \tilde{E}_{Tz}^* \right]$$

$$= \frac{1}{2\mu_2 \omega} \operatorname{Re}\underbrace{\left[i \left(|\tilde{\vec{E}}_T|^2 - 2 |\tilde{E}_{Tz}|^2 \right) \kappa \right]}_{\text{purely imaginary } \Rightarrow \operatorname{Re}[\cdot\cdot] = 0} = \underline{0}$$

The transmission coefficient is $T = \frac{I_T}{I_I}$

where $I_T = |\langle \vec{S}_T \rangle \cdot \hat{z}|$, $I_I = |\langle \vec{S}_I \rangle \cdot \hat{z}|$

Thus since $\langle \vec{S}_T \rangle \cdot \hat{z} = 0$, $I_T = 0 \Rightarrow \underline{\underline{T = 0}}$

We also see that $R + T = 1$, so energy is conserved, as expected.

Problem 3 (numbers are highly unofficial!)

(a) The skin depth is $d = 1/\kappa$ where

$$\kappa = \omega \sqrt{\frac{\epsilon\mu}{2}} \sqrt{\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1}$$

Poor conductor: $\sigma \ll \omega\epsilon \Rightarrow \sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2} \simeq 1 + \frac{1}{2}\left(\frac{\sigma}{\epsilon\omega}\right)^2$

$$\Rightarrow \kappa \simeq \omega \sqrt{\frac{\epsilon\mu}{2}} \frac{1}{\sqrt{2}} \frac{\sigma}{\epsilon\omega} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} \Rightarrow \underline{\underline{d \simeq \frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}}}$$

$$d = \frac{2}{\sigma}\sqrt{\frac{\epsilon_0\epsilon_r}{\mu_0\mu_r}} = \frac{2}{\sigma Z_0}\sqrt{\frac{\epsilon_r}{\mu_r}} \quad \text{where } Z_0 \equiv \sqrt{\frac{\mu_0}{\epsilon_0}} \simeq 376.7\,\Omega$$

("vacuum impedance")

Water: $\sigma \simeq 1/(2.5\cdot 10^5\,\Omega m)$, $\mu_r \simeq 1$, $\epsilon_r \simeq 1.8$ (at optical frequencies) $\Rightarrow \underline{\underline{d \simeq 1.8\cdot 10^3\ m}}$

(b) Good conductor: $\sigma \gg \omega\epsilon$

$$\Rightarrow \kappa \simeq k \simeq \omega\sqrt{\frac{\epsilon\mu}{2}}\sqrt{\frac{\sigma}{\epsilon\omega}} = \sqrt{\frac{\sigma\omega\mu}{2}}$$

$$\Rightarrow d = \frac{1}{\kappa} \simeq \frac{1}{k} = \underline{\underline{\frac{\lambda}{2\pi}}}$$

Typical metal: $\sigma \simeq 10^7\,(\Omega m)^{-1}$

$\mu \simeq \mu_0 = 4\pi\cdot 10^{-7}\ N/A^2$

Evaluate for visible range: $\omega \simeq 10^{15}\ s^{-1}$

$$\Rightarrow d = \sqrt{\frac{2}{10^7\cdot 10^{15}\cdot 4\pi\cdot 10^{-7}}}\ m \simeq 10^{-8}\ m = \underline{\underline{10\ nm}}$$

The very small skin depth implies rapid absorption of EM waves in metals $\Rightarrow$ light does not pass through. (Incident EM waves are also strongly reflected; see Sec. 9.4.2) $\Rightarrow$ opaque

(c) Phase difference $\phi = \arctan(\kappa/k) \simeq \arctan 1 = 45°$

Amplitude ratio :

$$\frac{B_0}{E_0} = \sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}} \approx \sqrt{\epsilon\mu\frac{\sigma}{\epsilon\omega}} = \sqrt{\frac{\sigma\mu}{\omega}}$$

Numerical example for typical metal :

$$\frac{B_0}{E_0} = \sqrt{\frac{10^7 \cdot 4\pi \cdot 10^{-7}}{10^{15}}} \sim 10^{-7}\,\frac{s}{m} \sim \frac{30}{c}$$

(the ratio is $1/c$ in vacuum)