Solution to tutorial 10, TFY 4240.

Problem 1

What we want is the "energy velocity" v_E, which can be defined as

$$v_E = \frac{\int da \, \langle \vec{S} \rangle \cdot \hat{z}}{\int da \, \langle u_{EM} \rangle} \qquad (*)$$

Here, the numerator is the power flow through a cross section of the wave guide (the flow direction is $\hat{z}$, so the cross section is $\perp$ to $\hat{z}$). The denominator is the EM energy density integrated over the same cross section. The cross section area is $\int da = A = L_x L_y$, where L_x and L_y are the lengths of the wave guide in the x- and y-directions. The expression $(*)$ is the analogue of the result

$$v = \frac{|\langle \vec{S} \rangle|}{u_{EM}} \qquad (**)$$

for a dielectric, in which the energy velocity v_E therefore equals the wave velocity $v = c/n$, which in this case is also the group velocity $v_g = \frac{d\omega}{dk} = \frac{d}{dk}(vk) = v$.

We will now show that $v_E = v_g$ holds for a hollow metallic TE waveguide as well. The need for the integral in $(*)$ (as opposed to in $(**)$) is due to the fact that in a waveguide quantities depend on the transverse coordinates x and y, unlike the propagation of a plane wave in a dielectric underlying (**).

Let us first find the energy density $\langle u_{EM}\rangle$:

$$\langle u_{EM}\rangle = \frac{1}{2}\left(\epsilon_0 \langle \vec{E}^2\rangle + \frac{1}{\mu_0}\langle \vec{B}^2\rangle\right)$$

$$= \frac{1}{4}\left(\epsilon_0 \tilde{\vec{E}}\cdot\tilde{\vec{E}}^* + \frac{1}{\mu_0}\tilde{\vec{B}}\cdot\tilde{\vec{B}}^*\right)$$

Since we are dealing with TE waves, $E_z = 0$, giving (using complex notation, see Griffiths' Problem 9.11; note that in $\langle u_{EM}\rangle$ the "Re" can be omitted since there are no phase differences)

$$\langle u_{EM}\rangle = \frac{1}{4}\left(\epsilon_0 \tilde{\vec{E}}_\perp\cdot\tilde{\vec{E}}_\perp^* + \frac{1}{\mu_0}\tilde{\vec{B}}_\perp\cdot\tilde{\vec{B}}_\perp^* + \frac{1}{\mu_0}\tilde{B}_z\tilde{B}_z^*\right)$$

$$= \frac{1}{4}\left(\epsilon_0 \vec{e}_\perp\cdot\vec{e}_\perp^* + \frac{1}{\mu_0}\vec{b}_\perp\cdot\vec{b}_\perp^* + \frac{1}{\mu_0} b_z b_z^*\right)$$

where, as in the lectures, $\vec{e}_i$, $\vec{b}_i$ stand for the complex amplitudes of $\tilde{\vec{E}}$ and $\tilde{\vec{B}}$.

We have (cf. Eqs. (9.180)(i) and (ii) in Griffiths)

$$e_x = \frac{i}{(\omega/c)^2 - k^2}\,\omega\frac{\partial b_z}{\partial y}, \quad e_y = \frac{-i}{(\omega/c)^2-k^2}\,\omega\frac{\partial b_z}{\partial x}$$

$$\Rightarrow \vec{e}_\perp\cdot\vec{e}_\perp^* = e_x e_x^* + e_y e_y^*$$

$$= \left(\frac{\omega}{(\omega/c)^2-k^2}\right)^2\left[\frac{\partial b_z}{\partial x}\frac{\partial b_z^*}{\partial x} + \frac{\partial b_z}{\partial y}\frac{\partial b_z^*}{\partial y}\right]$$

$$= \left(\frac{\omega}{(\omega/c)^2-k^2}\right)^2 (\nabla_\perp b_z)\cdot(\nabla_\perp b_z^*) \qquad \left(\nabla_\perp \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}\right)$$

Furthermore (cf. Eqs. (9.180) (iii) and (iv) in Griffiths)

$$b_x = \frac{i}{(\omega/c)^2 - k^2}\, k\, \frac{\partial b_z}{\partial x}, \qquad b_y = \frac{i}{(\omega/c)^2 - k^2}\, k\, \frac{\partial b_z}{\partial y}$$

$$\Rightarrow \quad \vec{b}_\perp \cdot \vec{b}_\perp^{\,*} = \left(\frac{k}{(\omega/c)^2 - k^2}\right)^2 \left[\frac{\partial b_z}{\partial x}\frac{\partial b_z^*}{\partial x} + \frac{\partial b_z}{\partial y}\frac{\partial b_z^*}{\partial y}\right]$$

$$= \left(\frac{k}{(\omega/c)^2 - k^2}\right)^2 \left(\nabla_\perp b_z\right)\cdot\left(\nabla_\perp b_z^*\right)$$

$$\Rightarrow \quad \epsilon_0\, \vec{e}_\perp \cdot \vec{e}_\perp^{\,*} + \frac{1}{\mu_0}\, \vec{b}_\perp \cdot \vec{b}_\perp^{\,*} \qquad \left(\text{use } \epsilon_0 \mu_0 = \frac{1}{c^2}\right)$$

$$= \frac{1}{\mu_0}\frac{(\omega/c)^2 + k^2}{\left[(\omega/c)^2 - k^2\right]^2} \left(\nabla_\perp b_z\right)\cdot\left(\nabla_\perp b_z^*\right)$$

Next, consider the cross section integral

$$\int da\, \left(\nabla_\perp b_z\right)\cdot\left(\nabla_\perp b_z^*\right) = \int_0^{L_x} dx \int_0^{L_y} dy \left[\frac{\partial b_z}{\partial x}\frac{\partial b_z^*}{\partial x} + \frac{\partial b_z}{\partial y}\frac{\partial b_z^*}{\partial y}\right]$$

$$= \int_0^{L_y} dy \int_0^{L_x} dx\, \frac{\partial b_z}{\partial x}\frac{\partial b_z^*}{\partial x} + \int_0^{L_x} dx \int_0^{L_y} dy\, \frac{\partial b_z}{\partial y}\frac{\partial b_z^*}{\partial y}$$

Doing an integration by parts gives

$$\int_0^{L_x} dx\, \frac{\partial b_z}{\partial x}\frac{\partial b_z^*}{\partial x} = \frac{\partial b_z}{\partial x}\, b_z^* \Big|_{x=0}^{x=L_x} - \int_0^{L_x} dx\, \frac{\partial^2 b_z}{\partial x^2}\, b_z^*$$

The boundary condition on $\vec{B}$ is $B_\perp = 0$. This implies $b_x = 0$ at $x=0$ and $x = L_x$, which in turn implies $\partial b_z/\partial x = 0$ there. Hence the surface term vanishes, giving

$$\int_0^{L_x} dx \, \frac{\partial b_z}{\partial x} \frac{\partial b_z^*}{\partial x} = - \int_0^{L_x} dx \, \frac{\partial^2 b_z}{\partial x^2} b_z^*$$

By similar reasoning ($B_\perp = 0$ implies $b_y = 0$ at $y = 0$ and $y = L_y$, which in turn implies $\partial b_z / \partial y = 0$ there) an integration by parts gives

$$\int_0^{L_y} dy \, \frac{\partial b_z}{\partial y} \frac{\partial b_z^*}{\partial y} = - \int_0^{L_y} dy \, \frac{\partial^2 b_z}{\partial y^2} b_z^*$$

Therefore

$$\int da \, (\nabla_\perp b_z) \cdot (\nabla_\perp b_z^*) = - \int da \, b_z^* \nabla_\perp^2 b_z$$

$$= \left[\left(\frac{\omega}{c}\right)^2 - k^2\right] \int da \, b_z^* b_z$$

where in the last transition we used the wave equation

$$\left[\nabla_\perp^2 + \left(\frac{\omega}{c}\right)^2 - k^2\right] b_z = 0.$$

Putting it all together, we find

$$\int da \, \langle u_{EM} \rangle = \frac{1}{4\mu_0} \left[\frac{(\omega/c)^2 + k^2}{(\omega/c)^2 - k^2} + 1 \right] \int da \, |b_z|^2$$

$$= \frac{1}{2\mu_0} \frac{(\omega/c)^2}{(\omega/c)^2 - k^2} \int da \, |b_z|^2$$

Next we consider the time-averaged Poynting vector

$$\langle \vec{S} \rangle = \frac{1}{2\mu_0} \mathrm{Re}\left(\tilde{\vec{E}} \times \tilde{\vec{B}}^* \right) = \frac{1}{2\mu_0} \mathrm{Re}\left(\vec{e} \times \vec{b}^* \right)$$

This gives

$$\langle \vec{S} \rangle \cdot \hat{z} = \langle S_z \rangle = \frac{1}{2\mu_0} \mathrm{Re}\left(\vec{e}_\perp \times \vec{b}_\perp^* \right)_z$$

since the longitudinal components e_z, b_z won't contribute to the z-component of $\vec{e} \times \vec{b}^*$.

$$\Rightarrow \left(\vec{e}_\perp \times \vec{b}_\perp^* \right)_z = e_x b_y^* - e_y b_x^*$$

$$= \frac{i}{(\omega/c)^2 - k^2} \omega \frac{\partial b_z}{\partial y} \frac{-i}{(\omega/c)^2 - k^2} k \frac{\partial b_z^*}{\partial y}$$

$$- \frac{i}{(\omega/c)^2 - k^2} (-\omega) \frac{\partial b_z}{\partial x} \frac{-i}{(\omega/c)^2 - k^2} k \frac{\partial b_z^*}{\partial x}$$

$$= \frac{\omega k}{\left[(\omega/c)^2 - k^2\right]^2} (\nabla_\perp b_z) \cdot (\nabla_\perp b_z^*) \qquad \text{(a real quantity)}$$

Thus (using our result for $\int da\, (\nabla_\perp b_z) \cdot (\nabla_\perp b_z^*)$)

$$\int da \langle \vec{S} \rangle \cdot \hat{z} = \frac{1}{2\mu_0} \frac{\omega k}{(\omega/c)^2 - k^2} \int da\, |b_z|^2$$

$$\Rightarrow v_E = \frac{\frac{1}{2\mu_0} \frac{\omega k}{(\omega/c)^2 - k^2} \int da\, |b_z|^2}{\frac{1}{2\mu_0} \frac{(\omega/c)^2}{(\omega/c)^2 - k^2} \int da\, |b_z|^2} = \underline{\underline{\frac{c^2 k}{\omega}}} \left(= \frac{c^2}{v_p} \right)$$

In lectures we found $\omega = \sqrt{(ck)^2 + \omega_{mn}^2}$

$$\Rightarrow \underline{\underline{v_g}} \equiv \frac{d\omega}{dk} = \frac{1}{2\omega} \cdot c^2 \cdot 2k = \frac{c^2 k}{\omega} = \underline{\underline{v_E}} \qquad \text{QED}$$

Problem 2

(a) $\vec{R} \equiv \vec{r} - \vec{r}\,' \Rightarrow R_i = r_i - r_i'$

$$\Rightarrow \frac{\partial}{\partial r_i} = \frac{\partial R_i}{\partial r_i}\frac{\partial}{\partial R_i} = 1 \cdot \frac{\partial}{\partial R_i} = \frac{\partial}{\partial R_i} \Rightarrow \nabla = \nabla_{\vec{R}}$$

Also, $\delta(\vec{r}-\vec{r}\,') = \delta(\vec{R})$, so $\nabla^2 G = -\delta(\vec{r}-\vec{r}\,')$ becomes $\nabla^2_{\vec{R}} G = -\delta(\vec{R})$. QED.

(b) Write $G(\vec{R}) = \frac{1}{(2\pi)^3}\int d^3k\; g(\vec{k})\, e^{i\vec{k}\cdot\vec{R}}$

which gives $\nabla^2_{\vec{R}} G(\vec{R}) = \frac{1}{(2\pi)^3}\int d^3k\; g(\vec{k})\, \nabla^2_{\vec{R}}\, e^{i\vec{k}\cdot\vec{R}}$

$$\nabla_{\vec{R}}\, e^{i\vec{k}\cdot\vec{R}} = i\vec{k}\, e^{i\vec{k}\cdot\vec{R}}$$

$$\Rightarrow \nabla^2_{\vec{R}}\, e^{i\vec{k}\cdot\vec{R}} = \nabla_{\vec{R}}\cdot\nabla_{\vec{R}}\, e^{i\vec{k}\cdot\vec{R}} = (i\vec{k})^2 e^{i\vec{k}\cdot\vec{R}} = -k^2 e^{i\vec{k}\cdot\vec{R}}$$

Using $\delta(\vec{R}) = \frac{1}{(2\pi)^3}\int d^3k\; e^{i\vec{k}\cdot\vec{R}}$ (Fourier representation of $\delta(\vec{R})$)

the equation $\nabla^2_{\vec{R}} G(\vec{R}) = -\delta(\vec{R})$ becomes

$$\frac{1}{(2\pi)^3}\int d^3k\; g(\vec{k})(-k^2)\, e^{i\vec{k}\cdot\vec{R}} = -\frac{1}{(2\pi)^3}\int d^3k\; e^{i\vec{k}\cdot\vec{R}}$$

Holds for all $\vec{R} \Rightarrow$ integrands identical $\Rightarrow \underline{\underline{g(\vec{k}) = 1/k^2}}$

(c) $G(\vec{R}) = \frac{1}{(2\pi)^3}\int d^3k\, \frac{1}{k^2}\, e^{i\vec{k}\cdot\vec{R}}$. Pick $\hat{z} = \hat{R} \Rightarrow \vec{k}\cdot\vec{R} = kR\cos\theta$

$$\Rightarrow G(\vec{R}) = \frac{1}{(2\pi)^3}\int_0^{2\pi} d\varphi \int_0^{\infty} dk\, k^2 \cdot \frac{1}{k^2}\int_{-1}^{1} d(\cos\theta)\, e^{ikR\cos\theta}$$

$$= \frac{1}{(2\pi)^2}\int_0^{\infty} dk\, \frac{1}{ikR}\, e^{ikR\cos\theta}\Big|_{-1}^{1} = \frac{1}{(2\pi)^2}\int_0^{\infty} dk\, \frac{1}{ikR}\underbrace{\left(e^{ikR} - e^{-ikR}\right)}_{2i\sin kR}$$

$$\underset{z \equiv kR}{=} \frac{2}{(2\pi)^2}\frac{1}{R}\underbrace{\int_0^{\infty} dz\, \frac{\sin z}{z}}_{\pi/2} = \underline{\underline{\frac{1}{4\pi R}}} \quad \text{QED}$$

Problem 3

(a) $$G(\vec{R},T) = \frac{1}{(2\pi)^4}\int d^3k \int d\omega \; g(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{R}-\omega T)}$$

$$\Rightarrow \left[\nabla^2_{\vec{R}} - \frac{1}{c^2}\frac{\partial^2}{\partial T^2}\right] G(\vec{R},T) = \frac{1}{(2\pi)^4}\int d^3k \int d\omega \; g(\vec{k},\omega)$$

$$\cdot \left[\nabla^2_{\vec{R}} - \frac{1}{c^2}\frac{\partial^2}{\partial T^2}\right] e^{i(\vec{k}\cdot\vec{R}-\omega T)}$$

Use $\nabla^2_{\vec{R}}\, e^{i\vec{k}\cdot\vec{R}} = -k^2\, e^{i\vec{k}\cdot\vec{R}}$ (see Problem 2)

and, similarly, $\frac{\partial^2}{\partial T^2} e^{-i\omega T} = (-i\omega)^2 e^{-i\omega T} = -\omega^2 e^{-i\omega T}$

$$\Rightarrow \left[\nabla^2_{\vec{R}} - \frac{1}{c^2}\frac{\partial^2}{\partial T^2}\right] G(\vec{R},T)$$

$$= -\frac{1}{(2\pi)^4}\int d^3k \int d\omega \; g(\vec{k},\omega)\left[k^2 - (\omega/c)^2\right]$$

On the other hand, we can write

$$\delta(\vec{R}) = \frac{1}{(2\pi)^3}\int d^3k \; e^{i\vec{k}\cdot\vec{R}}, \quad \delta(T) = \frac{1}{2\pi}\int d\omega \; e^{-i\omega T}$$

This gives

$$\int d^3k \int d\omega \left\{ g(\vec{k},\omega)\left[k^2 - (\omega/c)^2\right] - 1 \right\} e^{i\vec{k}\cdot\vec{R}}\, e^{-i\omega T} = 0$$

This holds for all $\vec{R}$ and all T $\Rightarrow$ integrand must vanish

$$\Rightarrow \quad \underline{\underline{g(\vec{k},\omega) = \frac{1}{k^2 - (\omega/c)^2}}}$$

(b) $$G(\vec{R},T) = \frac{1}{(2\pi)^4}\int d^3k \int d\omega \; \frac{1}{k^2-(\omega/c)^2}\, e^{i(\vec{k}\cdot\vec{R}-\omega T)}$$

Let us do the ω-integral first. Define

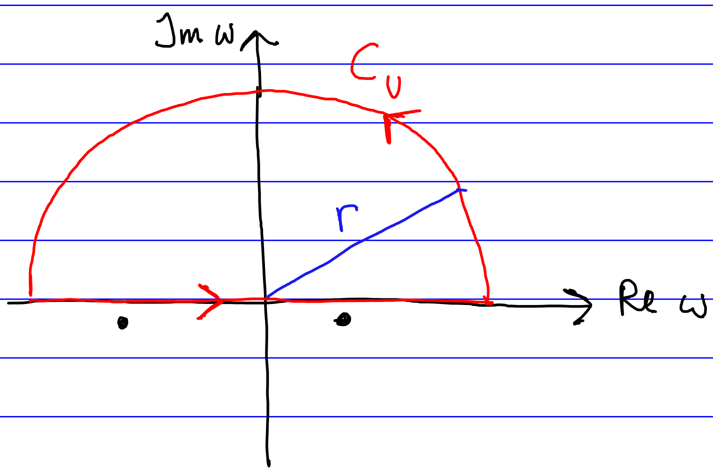
$$I \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega T}}{k^2-(\omega/c)^2}$$

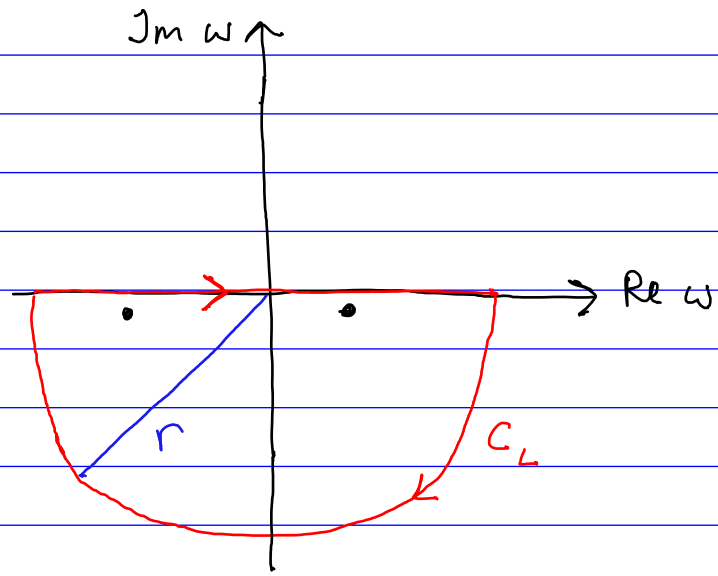
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega T}}{\left(k-\frac{\omega}{c}\right)\left(k+\frac{\omega}{c}\right)} = -\frac{c^2}{2\pi}\int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega T}}{(\omega-kc)(\omega+kc)}$$

The integrand has simple poles at $\omega = \pm kc$. We shift the poles infinitesimally down into the lower half plane by replacing $\omega \to \omega + i\eta$ where $\eta = 0^+$ (a positive infinitesimal)

$$\Rightarrow \quad I \to -\frac{c^2}{2\pi}\int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega T}}{(\omega-kc+i\eta)(\omega+kc+i\eta)}$$

(obviously the poles are now at $\omega = \pm kc - i\eta$, as desired). We will calculate the integral using contour integration. In addition to the integral along the real axis ($\int_{-\infty}^{\infty} d\omega \cdots$) we must then also integrate along a semicircle either in the upper or lower half plane, making a closed contour. The two possibilities (contours C_U or C_L) are shown in the figures below:





The poles at $\pm kc - i\eta$ are shown as black dots. (The actual contours have the radius $r \to \infty$.)

The choice of contour is based on the requirement that the contribution from the semicircle must vanish in the limit $r \to \infty$ due to decay from $e^{-i\omega T}$.

$$e^{-i\omega T} = e^{-i[\mathrm{Re}\,\omega + i\,\mathrm{Im}\,\omega]T} = e^{-i(\mathrm{Re}\,\omega)T}\, e^{(\mathrm{Im}\,\omega)T}$$

The decay must come from the factor $e^{(\mathrm{Im}\,\omega)T}$, since $e^{-i(\mathrm{Re}\,\omega)T}$ does not decay, it oscillates.

For $T < 0$ we need $\mathrm{Im}\,\omega > 0$ to make $e^{(\mathrm{Im}\,\omega)T} \to 0$ when $|\mathrm{Im}\,\omega| \to \infty$. So we must close the contour in the upper half plane, ie pick the contour C_U. Since there are no poles inside C_U, the residue theorem gives that the contour integral is 0, and thus also $I = 0$

$$\Rightarrow \; G(\vec{R}, T<0) = 0$$

(c) For $T > 0$ we need $\mathrm{Im}\,\omega < 0$ to make $e^{(\mathrm{Im}\,\omega)T} \to 0$ when $|\mathrm{Im}\,\omega| \to \infty$. So we must close the contour in the lower half plane, i.e. pick the contour C_L. The residue theorem then gives

$$I = -\frac{c^2}{2\pi}\cdot 2\pi i \cdot (-1) \left\{ \frac{e^{-i[kc - i\eta]T}}{(kc - i\eta) + kc + i\eta} + \frac{e^{-i[-kc - i\eta]T}}{(-kc - i\eta) - kc + i\eta} \right\}$$

$$= \frac{ic}{2k}\left(e^{-ikcT} - e^{ikcT} \right)$$

Residue theorem: $\oint_C f(z)\,dz = 2\pi i \sum_j \mathrm{Res}[f(z_j)]$ where $\{z_j\}$ are the singular points of f inside C

where in the last expression we let $\eta \to 0$ (as after the contour integration, the job of η was over). The factor (-1) was there because the contour C_L is clockwise.

$$\Rightarrow \quad G(\vec{R},T) = \frac{1}{(2\pi)^3} \int d^3k \; I \; e^{i\vec{k}\cdot\vec{R}}$$

$$= \frac{1}{(2\pi)^3} \int_0^{2\pi} d\varphi \int_0^{\infty} dk\, k^2 \int_{-1}^{1} d(\cos\theta)\; \frac{ic}{2k} \left(e^{-ikcT} - e^{ikcT} \right) e^{ikR\cos\theta}$$

The θ- and φ-integrals are the same as in Problem 2

$$\Rightarrow G(\vec{R},T) = \frac{1}{(2\pi)^2} \cdot \frac{ic}{2} \frac{1}{iR} \int_0^{\infty} dk \left(e^{-ikcT} - e^{ikcT} \right)\left(e^{ikR} - e^{-ikR} \right)$$

The integrand is even in k, so we can write $\int_0^{\infty} dk \cdots = \frac{1}{2}\int_{-\infty}^{\infty} dk \cdots$, giving

$$G(\vec{R},T) = \frac{c}{2R(2\pi)^2} \frac{1}{2} \int_{-\infty}^{\infty} dk \Big[e^{ik(R-cT)} - e^{ik(-R-cT)} - e^{ik(R+cT)} + e^{ik(-R+cT)} \Big]$$

Using $\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}$ (δ function in 1D) this gives

$$G(\vec{R},T) = \frac{c}{4R}\frac{1}{2\pi} \Big[\delta(R-cT) - \delta(-R-cT) - \delta(R+cT) + \delta(-R+cT) \Big]$$

$$= \frac{c}{4\pi R} \Big[\delta(cT-R) - \delta(cT+R) \Big]$$

where we used $\delta(-x) = \delta(x)$. Furthermore, since $T>0$, $cT+R>0$ (since $c>0$, $R\geq 0$) $\Rightarrow \delta(cT+R)=0$.
Also, using $\delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|}\delta(x-x_i)$ where $\{x_i\}$ are the zeros of $g(x)$, we can write $\delta(cT-R) \equiv \delta(g(T))$ with $g(T) = cT-R$, which has one zero, at $T = \frac{R}{c}$.
$g'(T) = c \Rightarrow g'(R/c) = c \Rightarrow \delta(cT-R) = \frac{1}{c}\delta(T-R/c)$

$$\Rightarrow \quad G(\vec{R},T) = \frac{1}{4\pi R}\,\delta\left(T - \frac{R}{c}\right) \qquad \text{Q.E.D.}$$

Problem 4

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3r' \, \frac{\rho(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|} \qquad (|\vec{r}-\vec{r}'| \equiv R)$$

$$\Rightarrow \frac{\partial V}{\partial t} = \frac{1}{4\pi\epsilon_0}\int d^3r' \, \frac{1}{R}\frac{\partial}{\partial t}\rho(\vec{r}',t_r)$$

$$t_r = t - \frac{R}{c} \Rightarrow \frac{\partial}{\partial t} = \frac{\partial t_r}{\partial t}\frac{\partial}{\partial t_r} = \frac{\partial}{\partial t_r}$$

$$\Rightarrow \underline{\frac{\partial V}{\partial t} = \frac{1}{4\pi\epsilon_0}\int d^3r' \, \frac{\dot{\rho}(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|}}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\vec{J}(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|}$$

$$\Rightarrow \nabla\cdot\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3r' \, \nabla\cdot\left(\frac{\vec{J}(\vec{r}',t_r)}{R}\right)$$

$$\nabla\cdot\left(\frac{\vec{J}}{R}\right) = \frac{1}{R}\nabla\cdot\vec{J} + \vec{J}\cdot\nabla\frac{1}{R}$$

On the other hand,

$$\nabla'\cdot\left(\frac{\vec{J}}{R}\right) = \frac{1}{R}\nabla'\cdot\vec{J} + \vec{J}\cdot\nabla'\frac{1}{R}$$

But $\nabla'\frac{1}{R} = -\nabla\frac{1}{R}$ (follows from $\frac{\partial R}{\partial x_i} = -\frac{\partial R}{\partial x_i'}$)

$$\Rightarrow \nabla\cdot\left(\frac{\vec{J}}{R}\right) = \frac{1}{R}\nabla\cdot\vec{J} - \vec{J}\cdot\nabla'\frac{1}{R}$$

$$= \frac{1}{R}\nabla\cdot\vec{J} + \frac{1}{R}(\nabla'\cdot\vec{J}) - \nabla'\cdot\left(\frac{\vec{J}}{R}\right)$$

Here $\nabla\cdot\vec{J}(\vec{r}',t_r) = \partial_i J_i(\vec{r}',t_r) = \dot{J}_i(\vec{r}',t_r)\,\partial_i t_r$

$$= \dot{J}_i\left(-\frac{1}{c}\right)\partial_i R = -\frac{1}{c}\,\dot{\vec{J}}\cdot\nabla R$$

Note that $\vec{J}$ depends on $\vec{r}'$ via both the first variable $\vec{r}'$ and the second variable t_r

$$\Rightarrow \nabla'\cdot\vec{J}(\vec{r}',t_r) = \left.\frac{\partial J_i(\vec{r}',t_r)}{\partial x_i'}\right|_{t_r} + \left.\frac{\partial J_i(\vec{r}',t_r)}{\partial t_r}\right|_{\vec{r}'}\frac{\partial t_r}{\partial x_i'}$$

The first term here equals $-\dot{\rho}(\vec{r}',t_r)$ (by the continuity equation). Furthermore,

$$\frac{\partial t_r}{\partial x_i'} = -\frac{1}{c}\frac{\partial R}{\partial x_i'} = \frac{1}{c}\frac{\partial R}{\partial x_i}$$

so that the second term becomes

$$\frac{1}{c}\dot{J}_i\frac{\partial R}{\partial x_i} = \frac{1}{c}\dot{\vec{J}}\cdot\nabla R$$

$$\Rightarrow \nabla\cdot\left(\frac{\vec{J}}{R}\right) = \frac{1}{R}\left(-\frac{1}{c}\right)\dot{\vec{J}}\cdot\nabla R + \frac{1}{R}\left[-\dot{\rho} + \frac{1}{c}\dot{\vec{J}}\cdot\nabla R\right]$$

$$-\nabla'\cdot\left(\frac{\vec{J}}{R}\right) = -\frac{\dot{\rho}}{R} - \nabla'\cdot\left(\frac{\vec{J}}{R}\right)$$

$$\Rightarrow \nabla\cdot\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = \frac{\mu_0}{4\pi}\int d^3r'\left[-\frac{\dot{\rho}}{R} - \nabla'\cdot\left(\frac{\vec{J}}{R}\right) + \overbrace{\frac{1}{c^2}\frac{1}{\epsilon_0\mu_0}}^{=1}\frac{\dot{\rho}}{R}\right]$$

$$= -\frac{\mu_0}{4\pi}\int d^3r'\,\nabla'\cdot\left(\frac{\vec{J}}{R}\right) \overset{\substack{\text{divergence}\\ \text{theorem}}}{=} -\frac{\mu_0}{4\pi}\int d\vec{a}'\cdot\frac{\vec{J}}{R} = \underline{\underline{0}} \quad \text{QED}$$

(the surface integral vanishes because the surface is "at infinity" where both $1/R$ and $\vec{J}$ vanish).