

Solution to tutorial 10, TFY 4240

Problem 1

What we want is the "energy velocity" v_E , which can be defined as

$$v_E = \frac{\int da \langle \vec{S} \rangle \cdot \hat{z}}{\int da \langle u_{EM} \rangle} \quad (*)$$

Here, the numerator is the power flow through a cross section of the wave guide (the flow direction is $\hat{z}$, so the cross section is $\perp$ to $\hat{z}$). The denominator is the EM energy density integrated over the same cross section. The cross section area is $\int da = A = L_x L_y$, where L_x and L_y are the lengths of the wave guide in the x - and y -directions. The expression (*) is the analogue of the result

$$v = \frac{|\langle \vec{S} \rangle|}{u_{EM}} \quad (**)$$

for a dielectric, in which the energy velocity v_E therefore equals the wave velocity $v = c/n$, which in this case is also the group velocity $v_g = \frac{d\omega}{dk} = \frac{d}{dk}(vk) = v$.

We will now show that $v_E = v_g$ holds for a hollow metallic TE waveguide as well. The need for the integral in (*) (as opposed to in (**)) is due to the fact that in a waveguide quantities depend on the transverse

coordinates x and y , unlike the propagation of a plane wave in a dielectric underlying (**).

Let us first find the energy density $\langle u_{EM} \rangle$:

$$\begin{aligned}\langle u_{EM} \rangle &= \frac{1}{2} (\epsilon_0 \langle \vec{E}^2 \rangle + \frac{1}{\mu_0} \langle \vec{B}^2 \rangle) \\ &= \frac{1}{4} (\epsilon_0 \vec{\tilde{E}} \cdot \vec{\tilde{E}}^* + \frac{1}{\mu_0} \vec{\tilde{B}} \cdot \vec{\tilde{B}}^*)\end{aligned}$$

Since we are dealing with TE waves, $E_z = 0$, giving (using complex notation, see Griffiths Problem 9.11; note that in $\langle u_{EM} \rangle$ the "Re" can be omitted since there are no phase differences)

$$\begin{aligned}\langle u_{EM} \rangle &= \frac{1}{4} (\epsilon_0 \vec{\tilde{E}}_{\perp} \cdot \vec{\tilde{E}}_{\perp}^* + \frac{1}{\mu_0} \vec{\tilde{B}}_{\perp} \cdot \vec{\tilde{B}}_{\perp}^* + \frac{1}{\mu_0} \tilde{B}_z \tilde{B}_z^*) \\ &= \frac{1}{4} (\epsilon_0 \vec{e}_{\perp} \cdot \vec{e}_{\perp}^* + \frac{1}{\mu_0} \vec{b}_{\perp} \cdot \vec{b}_{\perp}^* + \frac{1}{\mu_0} b_z b_z^*)\end{aligned}$$

where, as in the lectures, $\vec{e}_i, \vec{b}_i$ stand for the complex amplitudes of $\vec{\tilde{E}}$ and $\vec{\tilde{B}}$.

We have (cf. Eqs. (9.180)(i) and (ii) in Griffiths)

$$e_x = \frac{i}{(\omega/c)^2 - k^2} \omega \frac{\partial b_z}{\partial y}, \quad e_y = \frac{-i}{(\omega/c)^2 - k^2} \omega \frac{\partial b_z}{\partial x}$$

$$\Rightarrow \vec{e}_{\perp} \cdot \vec{e}_{\perp}^* = e_x e_x^* + e_y e_y^*$$

$$= \left(\frac{\omega}{(\omega/c)^2 - k^2} \right)^2 \left[\frac{\partial b_z}{\partial x} \frac{\partial b_z^*}{\partial x} + \frac{\partial b_z}{\partial y} \frac{\partial b_z^*}{\partial y} \right]$$

$$= \left(\frac{\omega}{(\omega/c)^2 - k^2} \right)^2 (\nabla_{\perp} b_z) \cdot (\nabla_{\perp} b_z^*) \quad (\nabla_{\perp} \equiv \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y})$$

Furthermore (cf. Eqs. (9.180) (iii) and (iv) in Griffiths)

$$b_x = \frac{i}{(\omega/c)^2 - k^2} k \frac{\partial b_z}{\partial x}, \quad b_y = \frac{i}{(\omega/c)^2 - k^2} k \frac{\partial b_z}{\partial y}$$

$$\Rightarrow \vec{b}_\perp \cdot \vec{b}_\perp^* = \left(\frac{k}{(\omega/c)^2 - k^2} \right)^2 \left[\frac{\partial b_z}{\partial x} \frac{\partial b_z^*}{\partial x} + \frac{\partial b_z}{\partial y} \frac{\partial b_z^*}{\partial y} \right]$$

$$= \left(\frac{k}{(\omega/c)^2 - k^2} \right)^2 (\nabla_\perp b_z) \cdot (\nabla_\perp b_z^*)$$

$$\Rightarrow \epsilon_0 \vec{e}_\perp \cdot \vec{e}_\perp^* + \frac{1}{\mu_0} \vec{b}_\perp \cdot \vec{b}_\perp^* \quad (\text{use } \epsilon_0 \mu_0 = \frac{1}{c^2})$$

$$= \frac{1}{\mu_0} \frac{(\omega/c)^2 + k^2}{[(\omega/c)^2 - k^2]^2} (\nabla_\perp b_z) \cdot (\nabla_\perp b_z^*)$$

Next, consider the cross section integral

$$\int da (\nabla_\perp b_z) \cdot (\nabla_\perp b_z^*) = \int_0^{L_x} dx \int_0^{L_y} dy \left[\frac{\partial b_z}{\partial x} \frac{\partial b_z^*}{\partial x} + \frac{\partial b_z}{\partial y} \frac{\partial b_z^*}{\partial y} \right]$$

$$= \int_0^{L_y} dy \int_0^{L_x} dx \frac{\partial b_z}{\partial x} \frac{\partial b_z^*}{\partial x} + \int_0^{L_x} dx \int_0^{L_y} dy \frac{\partial b_z}{\partial y} \frac{\partial b_z^*}{\partial y}$$

Doing an integration by parts gives

$$\int_0^{L_x} dx \frac{\partial b_z}{\partial x} \frac{\partial b_z^*}{\partial x} = \left. \frac{\partial b_z}{\partial x} b_z^* \right|_{x=0}^{x=L_x} - \int_0^{L_x} dx \frac{\partial^2 b_z}{\partial x^2} b_z^*$$

The boundary condition on $\vec{B}$ is $B_\perp = 0$. This implies $b_x = 0$ at $x=0$ and $x=L_x$, which in turn implies $\partial b_z / \partial x = 0$ there. Hence the

surface term vanishes, giving

$$\int_0^{L_x} dx \frac{\partial b_z}{\partial x} \frac{\partial b_z^*}{\partial x} = - \int_0^{L_x} dx \frac{\partial^2 b_z}{\partial x^2} b_z^*$$

By similar reasoning ($B_z = 0$ implies $b_y = 0$ at $y=0$ and $y=L_y$, which in turn implies $\partial b_z / \partial y = 0$ there) an integration by parts gives

$$\int_0^{L_y} dy \frac{\partial b_z}{\partial y} \frac{\partial b_z^*}{\partial y} = - \int_0^{L_y} dy \frac{\partial^2 b_z}{\partial y^2} b_z^*$$

Therefore

$$\begin{aligned} \int da (\nabla_{\perp} b_z) \cdot (\nabla_{\perp} b_z^*) &= - \int da b_z^* \nabla_{\perp}^2 b_z \\ &= \left[\left(\frac{\omega}{c} \right)^2 - k^2 \right] \int da b_z^* b_z \end{aligned}$$

where in the last transition we used the wave equation

$$\left[\nabla_{\perp}^2 + \left(\frac{\omega}{c} \right)^2 - k^2 \right] b_z = 0.$$

Pulling it all together, we find

$$\int da \langle u_{EM} \rangle = \frac{1}{4\mu_0} \left[\frac{(\omega/c)^2 + k^2}{(\omega/c)^2 - k^2} + 1 \right] \int da |b_z|^2$$

$$= \frac{1}{2\mu_0} \frac{(\omega/c)^2}{(\omega/c)^2 - k^2} \int da |b_z|^2$$

Next we consider the time-averaged Poynting vector

$$\langle \vec{S} \rangle = \frac{1}{2\mu_0} \operatorname{Re} (\vec{\tilde{E}} \times \vec{\tilde{B}}^*) = \frac{1}{2\mu_0} \operatorname{Re} (\vec{e} \times \vec{b}^*)$$

This gives

$$\langle \vec{S} \rangle \cdot \hat{z} = \langle S_z \rangle = \frac{1}{2\mu_0} \operatorname{Re} (\vec{e}_\perp \times \vec{b}_\perp^*)_z$$

since the longitudinal components e_z, b_z won't contribute to the z -component of $\vec{e} \times \vec{b}^*$.

$$\Rightarrow (\vec{e}_\perp \times \vec{b}_\perp^*)_z = e_x b_y^* - e_y b_x^*$$

$$= \frac{i}{(\omega/c)^2 - k^2} \omega \frac{\partial b_z}{\partial y} \frac{-i}{(\omega/c)^2 - k^2} k \frac{\partial b_z^*}{\partial y}$$

$$- \frac{i}{(\omega/c)^2 - k^2} (-\omega) \frac{\partial b_z}{\partial x} \frac{-i}{(\omega/c)^2 - k^2} k \frac{\partial b_z^*}{\partial x}$$

$$= \frac{\omega k}{[(\omega/c)^2 - k^2]^2} (\nabla_\perp b_z) \cdot (\nabla_\perp b_z^*) \quad (\text{a real quantity})$$

Thus (using our result for $\int da (\nabla_\perp b_z) \cdot (\nabla_\perp b_z^*)$)

$$\int da \langle \vec{S} \rangle \cdot \hat{z} = \frac{1}{2\mu_0} \frac{\omega k}{(\omega/c)^2 - k^2} \int da |b_z|^2$$

$$\Rightarrow \underline{v_E} = \frac{\frac{1}{2\mu_0} \frac{\omega k}{(\omega/c)^2 - k^2} \int da |b_z|^2}{\frac{1}{2\mu_0} \frac{(\omega/c)^2}{(\omega/c)^2 - k^2} \int da |b_z|^2} = \underline{\underline{\frac{c^2 k}{\omega}}} \left(= \frac{c^2}{v_p} \right)$$

In lecture we found $\omega = \sqrt{(ck)^2 + \omega_{\text{min}}^2}$

$$\Rightarrow \underline{\underline{v_g}} \equiv \frac{d\omega}{dk} = \frac{1}{2\omega} \cdot c^2 \cdot 2k = \frac{c^2 k}{\omega} = \underline{\underline{v_E}} \quad \text{QED}$$

Problem 2

$$(a) \quad \vec{R} \equiv \vec{r} - \vec{r}' \Rightarrow R_i = r_i - r'_i$$

$$\Rightarrow \frac{\partial}{\partial r_i} = \frac{\partial R_i}{\partial r_i} \frac{\partial}{\partial R_i} = 1 \cdot \frac{\partial}{\partial R_i} = \frac{\partial}{\partial R_i} \Rightarrow \nabla^2 = \nabla_{\vec{R}}^2$$

Also, $\delta(\vec{r} - \vec{r}') = \delta(\vec{R})$, so $\nabla^2 G = -\delta(\vec{r} - \vec{r}')$ becomes $\nabla_{\vec{R}}^2 G = -\delta(\vec{R})$. QED.

$$(b) \quad \text{Write } G(\vec{R}) = \frac{1}{(2\pi)^3} \int d^3k \, g(\vec{k}) e^{i\vec{k} \cdot \vec{R}}$$

$$\text{which gives } \nabla_{\vec{R}}^2 G(\vec{R}) = \frac{1}{(2\pi)^3} \int d^3k \, g(\vec{k}) \nabla_{\vec{R}}^2 e^{i\vec{k} \cdot \vec{R}}$$

$$\nabla_{\vec{R}} e^{i\vec{k} \cdot \vec{R}} = i\vec{k} e^{i\vec{k} \cdot \vec{R}}$$

$$\Rightarrow \nabla_{\vec{R}}^2 e^{i\vec{k} \cdot \vec{R}} = \nabla_{\vec{R}} \cdot \nabla_{\vec{R}} e^{i\vec{k} \cdot \vec{R}} = (i\vec{k})^2 e^{i\vec{k} \cdot \vec{R}} = -k^2 e^{i\vec{k} \cdot \vec{R}}$$

$$\text{Using } \delta(\vec{R}) = \frac{1}{(2\pi)^3} \int d^3k \, e^{i\vec{k} \cdot \vec{R}} \quad (\text{Fourier representation of } \delta(\vec{R}))$$

the equation $\nabla_{\vec{R}}^2 G(\vec{R}) = -\delta(\vec{R})$ becomes

$$\frac{1}{(2\pi)^3} \int d^3k \, g(\vec{k}) (-k^2) e^{i\vec{k} \cdot \vec{R}} = -\frac{1}{(2\pi)^3} \int d^3k \, e^{i\vec{k} \cdot \vec{R}}$$

Holds for all $\vec{R} \Rightarrow$ integrands identical $\Rightarrow \underline{g(\vec{k}) = 1/k^2}$

$$(c) \quad G(\vec{R}) = \frac{1}{(2\pi)^3} \int d^3k \, \frac{1}{k^2} e^{i\vec{k} \cdot \vec{R}} \quad \text{Pick } \hat{z} = \hat{R} \Rightarrow \vec{k} \cdot \vec{R} = kR \cos\theta$$

$$\Rightarrow G(\vec{R}) = \frac{1}{(2\pi)^3} \int_0^{2\pi} d\phi \int_0^\infty dk \, k^2 \cdot \frac{1}{k^2} \int_{-1}^1 d(\cos\theta) e^{ikR \cos\theta}$$

$$= \frac{1}{(2\pi)^2} \int_0^\infty dk \, \frac{1}{ikR} e^{ikR \cos\theta} \Big|_{-1}^1 = \frac{1}{(2\pi)^2} \int_0^\infty dk \, \frac{1}{ikR} \underbrace{\left(e^{ikR} - e^{-ikR} \right)}_{2i \sin kR}$$

$$\stackrel{\vec{R} = z\hat{z}}{=} \frac{2}{(2\pi)^2} \frac{1}{R} \underbrace{\int_0^\infty dz \, \frac{\sin z}{z}}_{\pi/2} = \underline{\underline{\frac{1}{4\pi R}}} \quad \text{QED}$$

Problem 3

$$(a) \quad G(\vec{R}, T) = \frac{1}{(2\pi)^4} \int d^3k \int d\omega \, g(\vec{k}, \omega) e^{i(\vec{k} \cdot \vec{R} - \omega T)}$$

$$\Rightarrow \left[\nabla_{\vec{R}}^2 - \frac{1}{c^2} \frac{\partial^2}{\partial T^2} \right] G(\vec{R}, T) = \frac{1}{(2\pi)^4} \int d^3k \int d\omega \, g(\vec{k}, \omega)$$

$$\cdot \left[\nabla_{\vec{R}}^2 - \frac{1}{c^2} \frac{\partial^2}{\partial T^2} \right] e^{i(\vec{k} \cdot \vec{R} - \omega T)}$$

$$\text{Use } \nabla_{\vec{R}}^2 e^{i\vec{k} \cdot \vec{R}} = -k^2 e^{i\vec{k} \cdot \vec{R}} \quad (\text{see Problem 2})$$

$$\text{and, similarly, } \frac{\partial^2}{\partial T^2} e^{-i\omega T} = (-i\omega)^2 e^{-i\omega T} = -\omega^2 e^{-i\omega T}$$

$$\Rightarrow \left[\nabla_{\vec{R}}^2 - \frac{1}{c^2} \frac{\partial^2}{\partial T^2} \right] G(\vec{R}, T)$$

$$= -\frac{1}{(2\pi)^4} \int d^3k \int d\omega \, g(\vec{k}, \omega) [k^2 - (\omega/c)^2]$$

On the other hand, we can write

$$\delta(\vec{R}) = \frac{1}{(2\pi)^3} \int d^3k \, e^{i\vec{k} \cdot \vec{R}}, \quad \delta(T) = \frac{1}{2\pi} \int d\omega \, e^{-i\omega T}$$

This gives

$$\int d^3k \int d\omega \, \{ g(\vec{k}, \omega) [k^2 - (\omega/c)^2] - 1 \} e^{i\vec{k} \cdot \vec{R}} e^{-i\omega T} = 0$$

This holds for all $\vec{R}$ and all $T \Rightarrow$ integrand must vanish

$$\Rightarrow \underline{\underline{g(\vec{k}, \omega) = \frac{1}{k^2 - (\omega/c)^2}}}$$

$$(b) \quad G(\vec{R}, T) = \frac{1}{(2\pi)^4} \int d^3k \int d\omega \, \frac{1}{k^2 - (\omega/c)^2} e^{i(\vec{k} \cdot \vec{R} - \omega T)}$$

Let us do the ω -integral first. Define

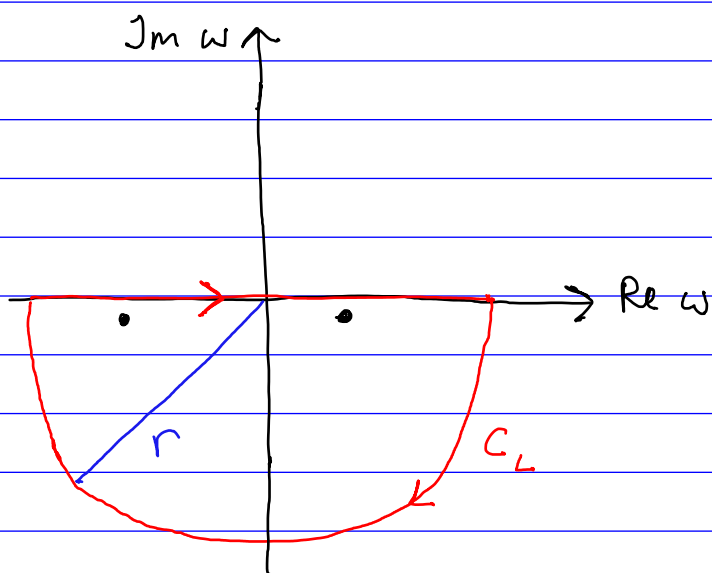
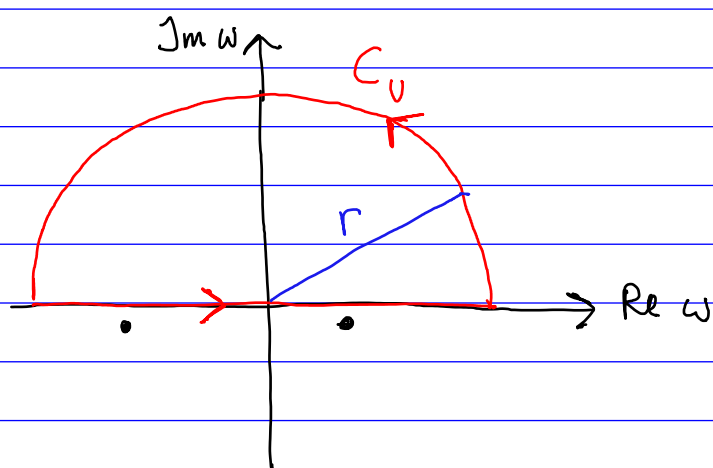
$$I \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega T}}{k^2 - (\omega/c)^2}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega T}}{(k - \frac{\omega}{c})(k + \frac{\omega}{c})} = -\frac{c^2}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega T}}{(\omega - kc)(\omega + kc)}$$

The integrand has simple poles at $\omega = \pm kc$. We shift the poles infinitesimally down into the lower half plane by replacing $\omega \rightarrow \omega + i\eta$ where $\eta = 0^+$ (a positive infinitesimal)

$$\Rightarrow I \rightarrow -\frac{c^2}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega T}}{(\omega - kc + i\eta)(\omega + kc + i\eta)}$$

(obviously the poles are now at $\omega = \pm kc - i\eta$, as desired). We will calculate the integral using contour integration. In addition to the integral along the real axis ($\int_{-\infty}^{\infty} d\omega \dots$) we must then also integrate along a $-\infty$ semicircle either in the upper or lower half plane, making a closed contour. The two possibilities (contours C_U or C_L) are shown in the figures below:



The poles at $\pm kc - i\eta$ are shown as black dots. (The actual contours have the radius $r \rightarrow \infty$.)

The choice of contour is based on the requirement that the contribution from the semicircle must vanish in the limit $r \rightarrow \infty$ due to decay from $e^{-i\omega T}$.

$$e^{-i\omega T} = e^{-i[\operatorname{Re}\omega + i\operatorname{Im}\omega]T} = e^{-i(\operatorname{Re}\omega)T} e^{(\operatorname{Im}\omega)T}$$

The decay must come from the factor $e^{(\operatorname{Im}\omega)T}$, since $e^{-i(\operatorname{Re}\omega)T}$ does not decay, it oscillates.

For $T < 0$ we need $\operatorname{Im}\omega > 0$ to make $e^{(\operatorname{Im}\omega)T} \rightarrow 0$ when $|\operatorname{Im}\omega| \rightarrow \infty$. So we must close the contour in the upper half plane, i.e. pick the contour C_U . Since there are no poles inside C_U , the residue theorem gives that the contour integral is 0, and thus also $I = 0$

$$\Rightarrow G(\vec{R}, T < 0) = 0$$

(c) For $T > 0$ we need $\operatorname{Im}\omega < 0$ to make $e^{(\operatorname{Im}\omega)T} \rightarrow 0$ when $|\operatorname{Im}\omega| \rightarrow \infty$. So we must close the contour in the lower half plane, i.e. pick the contour C_L . The residue theorem then gives

$$I = -\frac{c^2}{2\pi} \cdot 2\pi i \cdot (-1) \left\{ \frac{e^{-i[kc - i\eta]T}}{(kc - i\eta) + kc + i\eta} + \frac{e^{-i[-kc - i\eta]T}}{(-kc - i\eta) - kc + i\eta} \right\}$$

$$= \frac{ic}{2k} \left(e^{-ikcT} - e^{ikcT} \right)$$

Residue theorem: $\oint f(z) dz = 2\pi i \sum_j \operatorname{Res}[f(z_j)]$ where $\{z_j\}$ are the singular points of f inside C

where in the last expression we let $\eta \rightarrow 0$ (as after the contour integration, the job of η was over). The factor (-1) was there because the contour C_L is clockwise.

$$\Rightarrow G(\vec{R}, T) = \frac{1}{(2\pi)^3} \int d^3k \, I \, e^{i\vec{k} \cdot \vec{R}}$$

$$= \frac{1}{(2\pi)^3} \int_0^{2\pi} d\varphi \int_0^\infty dk \, k^2 \int_{-1}^1 d(\cos\theta) \, \frac{ic}{2k} \left(e^{-ikcT} - e^{ikcT} \right) e^{ikR \cos\theta}$$

The θ - and φ -integrals are the same as in Problem 2

$$\Rightarrow G(\vec{R}, T) = \frac{1}{(2\pi)^2} \cdot \frac{ic}{2} \frac{1}{iR} \int_0^\infty dk \left(e^{-ikcT} - e^{ikcT} \right) \left(e^{ikR} - e^{-ikR} \right)$$

The integrand is even in k , so we can write

$$\int_0^\infty dk \dots = \frac{1}{2} \int_{-\infty}^\infty dk \dots, \text{ giving}$$

$$G(\vec{R}, T) = \frac{c}{2R(2\pi)^2} \frac{1}{2} \int_{-\infty}^\infty dk \left[e^{ik(R-cT)} - e^{ik(-R-cT)} - e^{ik(R+cT)} + e^{ik(-R+cT)} \right]$$

Using $\delta(x) = \frac{1}{2\pi} \int_{-\infty}^\infty dk \, e^{ikx}$ (δ function in 1D) this gives

$$G(\vec{R}, T) = \frac{c}{4R} \frac{1}{2\pi} \left[\delta(R-cT) - \delta(-R-cT) - \delta(R+cT) + \delta(-R+cT) \right]$$

$$= \frac{c}{4\pi R} \left[\delta(cT-R) - \delta(cT+R) \right]$$

where we used $\delta(-x) = \delta(x)$. Furthermore, since $T > 0$, $cT + R > 0$ (since $c > 0, R \geq 0$) $\Rightarrow \delta(cT+R) = 0$.

Also, using $\delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|} \delta(x-x_i)$ where $\{x_i\}$ are the zeros of $g(x)$, we can write $\delta(cT-R) \equiv \delta(g(T))$

with $g(T) = cT - R$, which has one zero, at $T = \frac{R}{c}$.

$$g'(T) = c \Rightarrow g'(R/c) = c \Rightarrow \delta(cT-R) = \frac{1}{c} \delta(T - R/c)$$

$$\Rightarrow G(\vec{R}, T) = \frac{1}{4\pi R} \delta\left(T - \frac{R}{c}\right) \quad \text{Q.E.D.}$$

Problem 4 (numbers are highly unofficial!)

(a) The skin depth is $d = 1/\kappa$ where

$$\kappa = \omega \sqrt{\frac{\epsilon\mu}{2}} \sqrt{\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1}$$

Poor conductor: $\sigma \ll \omega\epsilon \Rightarrow \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} \approx 1 + \frac{1}{2}\left(\frac{\sigma}{\epsilon\omega}\right)^2$

$$\Rightarrow \kappa \approx \omega \sqrt{\frac{\epsilon\mu}{2}} \frac{1}{\sqrt{2}} \frac{\sigma}{\epsilon\omega} = \frac{\sigma}{2} \sqrt{\frac{\mu}{\epsilon}} \Rightarrow \underline{\underline{d \approx \frac{2}{\sigma} \sqrt{\frac{\epsilon}{\mu}}}}$$

$$d = \frac{2}{\sigma} \sqrt{\frac{\epsilon_0 \epsilon_r}{\mu_0 \mu_r}} = \frac{2}{\sigma Z_0} \sqrt{\frac{\epsilon_r}{\mu_r}} \quad \text{where } Z_0 \equiv \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 376.7 \Omega$$

("vacuum impedance")

Water: $\sigma \approx 1/(2.5 \cdot 10^5 \Omega\text{m})$, $\mu_r \approx 1$, $\epsilon_r \approx 1.8$ (at optical frequencies) $\Rightarrow \underline{\underline{d \approx 1.8 \cdot 10^3 \text{ m}}}$

(b) Good conductor: $\sigma \gg \omega\epsilon$

$$\Rightarrow \kappa \approx k \approx \omega \sqrt{\frac{\epsilon\mu}{2}} \sqrt{\frac{\sigma}{\epsilon\omega}} = \sqrt{\frac{\sigma\omega\mu}{2}}$$

$$\Rightarrow d = \frac{1}{\kappa} \approx \frac{1}{k} = \underline{\underline{\frac{\lambda}{2\pi}}}$$

Typical metal: $\sigma \approx 10^7 (\Omega\text{m})^{-1}$
 $\mu \approx \mu_0 = 4\pi \cdot 10^{-7} \text{ N/A}^2$

Evaluate for visible range: $\omega \approx 10^{15} \text{ s}^{-1}$

$$\Rightarrow d = \sqrt{\frac{2}{10^7 \cdot 10^{15} \cdot 4\pi \cdot 10^{-7}}} \text{ m} \approx 10^{-8} \text{ m} = \underline{\underline{10 \text{ nm}}}$$

The very small skin depth implies rapid absorption of EM waves in metals $\Rightarrow$ light does not pass through. (Incident EM waves are also strongly reflected; see Sec. 9.4.2) $\Rightarrow$ opaque

(c) Phase difference $\phi = \arctan(\kappa/k) \approx \arctan 1 = 45^\circ$

Amplitude ratio :

$$\frac{B_0}{E_0} = \sqrt{\epsilon\mu} \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} \approx \sqrt{\epsilon\mu} \frac{\sigma}{\epsilon\omega} = \sqrt{\frac{\sigma\mu}{\omega}}$$

Numerical example for typical metal :

$$\frac{B_0}{E_0} = \sqrt{\frac{10^7 \cdot 4\pi \cdot 10^{-7}}{10^{15}}} \sim 10^{-7} \frac{\text{s}}{\text{m}} \sim \frac{30}{c}$$

(the ratio is $1/c$ in vacuum)