Solution to tutorial 11, TFY4240

Problem 1

The Jefimenko equation for $\vec{B}$ is

$$\vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3r' \left[\frac{\vec{J}(\vec{r}',t_r)}{R^2} + \frac{\dot{\vec{J}}(\vec{r}',t_r)}{cR} \right] \times \hat{R}$$

where $\vec{R} = \vec{r} - \vec{r}'$

If we neglect higher time derivatives than $\dot{\vec{J}}$, we get, upon expanding $\vec{J}$ and $\dot{\vec{J}}$ around time t,

$$\vec{J}(\vec{r}',t_r) \approx \vec{J}(\vec{r}',t) + (t_r - t)\,\dot{\vec{J}}(\vec{r}',t)$$

$$\dot{\vec{J}}(\vec{r}',t_r) \approx \dot{\vec{J}}(\vec{r}',t)$$

This gives

$$\vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3r'$$

$$\left\{ \frac{\vec{J}(\vec{r}',t)}{R^2} + \underbrace{\left(\frac{t_r - t}{R} + \frac{1}{c} \right)}_{=0} \frac{\dot{\vec{J}}(\vec{r}',t)}{R} \right\} \times \hat{R}$$

$$= \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\vec{J}(\vec{r}',t)}{R^2} \times \hat{R}$$

since $t_r = t - \frac{R}{c} \Rightarrow \frac{t_r - t}{R} = -\frac{1}{c}$

Problem 2

The Liénard-Wiechert expression for $\vec{A}$ is

$$\vec{A}(\vec{r},t) = \frac{\vec{v}}{c^2} V(\vec{r},t) = \frac{\mu_0}{4\pi} \frac{qc\vec{v}}{Rc - \vec{R}\cdot\vec{v}}$$

where $\vec{R}$, R and $\vec{v}$ are all evaluated at the retarded time t_r.

$$\frac{\partial \vec{A}(\vec{r},t)}{\partial t} = \frac{\mu_0 qc}{4\pi}\left[\frac{1}{Rc - \vec{R}\cdot\vec{v}} \frac{\partial \vec{v}}{\partial t} + \vec{v}\frac{\partial}{\partial t}\frac{1}{Rc - \vec{R}\cdot\vec{v}}\right]$$

$$\underline{\frac{\partial \vec{v}}{\partial t}} = \frac{\partial t_r}{\partial t}\frac{d\vec{v}}{dt_r} = \underline{\frac{\partial t_r}{\partial t}\vec{a}}$$

$$\frac{\partial}{\partial t}\frac{1}{Rc - \vec{R}\cdot\vec{v}} = -\frac{1}{(Rc - \vec{R}\cdot\vec{v})^2}\frac{\partial}{\partial t}(Rc - \vec{R}\cdot\vec{v})$$

$$\frac{\partial}{\partial t}R = \frac{\partial t_r}{\partial t}\frac{dR}{dt_r}$$

$$\frac{dR}{dt_r} = \frac{d}{dt_r}\sqrt{\vec{R}^2} = \frac{1}{2\sqrt{\vec{R}^2}}\cdot 2\vec{R}\cdot\frac{d\vec{R}}{dt_r} = \hat{R}\cdot\frac{-d\vec{r}_q}{dt_r} = -\hat{R}\cdot\vec{v}$$

$$\frac{\partial}{\partial t}\vec{R}\cdot\vec{v} = \frac{\partial \vec{R}}{\partial t}\cdot\vec{v} + \vec{R}\cdot\frac{\partial \vec{v}}{\partial t}$$

$$\frac{\partial \vec{R}}{\partial t} = \frac{\partial t_r}{\partial t}\frac{d\vec{R}}{dt_r} = -\frac{\partial t_r}{\partial t}\vec{v}$$

$$\Rightarrow \underline{\frac{\partial}{\partial t}\frac{1}{Rc - \vec{R}\cdot\vec{v}} = -\frac{\frac{\partial t_r}{\partial t}}{(Rc - \vec{R}\cdot\vec{v})^2}\left[-c\hat{R}\cdot\vec{v} + v^2 - \vec{R}\cdot\vec{a}\right]}$$

$$\Rightarrow \frac{\partial \vec{A}}{\partial t} = \frac{\mu_0 q c}{4\pi} \frac{\frac{\partial t_r}{\partial t}}{Rc - \vec{R}\cdot\vec{v}} \left[\vec{a} + \vec{v}\, \frac{c\hat{R}\cdot\vec{v} - v^2 + \vec{R}\cdot\vec{a}}{Rc - \vec{R}\cdot\vec{v}} \right]$$

$$= \frac{\mu_0 q c}{4\pi} \frac{(\partial t_r/\partial t)}{(Rc - \vec{R}\cdot\vec{v})^2} \left[cR\vec{a} - (\vec{R}\cdot\vec{v})\vec{a} + c\vec{v}(\hat{R}\cdot\vec{v}) - v^2\vec{v} + (\vec{R}\cdot\vec{a})\vec{v} \right]$$

It remains to find $\partial t_r/\partial t$. From $t_r = t - \frac{R(t_r)}{c}$ we get

$$\frac{\partial t_r}{\partial t} = 1 - \frac{1}{c}\frac{\partial}{\partial t} R(t_r) = 1 - \frac{1}{c}\frac{\partial t_r}{\partial t}\frac{dR}{dt_r} = 1 - \frac{1}{c}\frac{\partial t_r}{\partial t}(-\hat{R}\cdot\vec{v})$$

i.e. $$\frac{\partial t_r}{\partial t} = 1 + \frac{\hat{R}\cdot\vec{v}}{c}\frac{\partial t_r}{\partial t}$$

Solving this equation for $\frac{\partial t_r}{\partial t}$ gives

$$\underline{\frac{\partial t_r}{\partial t}} = \frac{1}{1 - \hat{R}\cdot\vec{v}/c} = \frac{Rc}{Rc - \vec{R}\cdot\vec{v}} = \underline{\frac{Rc}{\vec{R}\cdot\vec{u}}}$$ (cf. Eq. (10.71) in Griffiths)

since $\vec{R}\cdot\vec{u} = \vec{R}\cdot(c\hat{R} - \vec{v}) = Rc - \vec{R}\cdot\vec{v}$.

Inserting the expression for $\frac{\partial t_r}{\partial t}$ into $\frac{\partial \vec{A}}{\partial t}$ gives

$$\frac{\partial \vec{A}(\vec{r},t)}{\partial t} = \frac{\mu_0 q c}{4\pi} \frac{Rc}{(Rc - \vec{R}\cdot\vec{v})^3} \left[cR\vec{a} - (\vec{R}\cdot\vec{v})\vec{a} + c\vec{v}(\hat{R}\cdot\vec{v}) - v^2\vec{v} + (\vec{R}\cdot\vec{a})\vec{v} \right]$$

$$= \frac{qR}{4\pi(Rc - \vec{R}\cdot\vec{v})^3} \left[(cR - \vec{R}\cdot\vec{v})\vec{a} + (c\hat{R}\cdot\vec{v} + \vec{R}\cdot\vec{a} - v^2)\vec{v} \right]$$

which agrees with Eq. (10.63) in Griffiths (in which the terms $\pm Rc\vec{v}$ inside the square brackets cancel))

A remark:

The expression we found for $\partial t_r/\partial t$,

$$\frac{\partial t_r}{\partial t} = \frac{1}{1 - \frac{\vec{v}}{c}\cdot\hat{R}}$$

is the inverse of the expression we found in the lectures for dt/dt_r,

$$\frac{dt}{dt_r} = 1 - \frac{\vec{v}}{c}\cdot\hat{R}$$

(discussed in connection with a relativistically valid expression for radiation from a point charge). This is not a coincidence. Exactly the same variables are held constant while carrying out these two differentiations, so these two derivatives should then be each other's inverses. (The retarded time t_r is a function of the "observation time" t as well as of the "observation point" $\vec{r}$, and $\partial t_r/\partial t$ means $\left.\frac{\partial t_r}{\partial t}\right|_{\vec{r}}$, i.e. with the observation point $\vec{r}$ held fixed. $\vec{r}$ is fixed also when calculating dt/dt_r, but as t and $\vec{r}$ are variables that we choose independently, t is not a function of $\vec{r}$, so in dt/dt_r we use an ordinary derivative "d", not a partial derivative "∂".)

Problem 3

The point charge moves with constant velocity $\vec{v}$
$\Rightarrow$ $\vec{r}_q(t') = \vec{v}\,t'$ (without any loss of generality we have taken $\vec{r}_q = 0$ at $t' = 0$)

The acceleration is zero $\Rightarrow$ the electric field at $(\vec{r}, t)$ is

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{R(t_r)\,\vec{u}(t_r)}{\vec{R}(t_r)\cdot\vec{u}(t_r)}(c^2 - v^2)$$

(where we have explicitly indicated that $\vec{R}$ and $\vec{u}$ are here to be evaluated at the retarded time t_r).
We will now rewrite $R(t_r)\,\vec{u}(t_r)$ and $\vec{R}(t_r)\cdot\vec{u}(t_r)$:

$$\underline{R(t_r)\,\vec{u}(t_r)} = R(t_r)\left(c\hat{R}(t_r) - \vec{v}\right)$$

$$= c\vec{R}(t_r) - R(t_r)\vec{v} = c\left(\vec{r} - \vec{r}_q(t_r)\right) - c(t - t_r)\vec{v}$$

$$= c\vec{r} - c\vec{v}t_r - ct\vec{v} + c\vec{v}t_r = c(\vec{r} - \vec{v}t)$$

$$= c(\vec{r} - \vec{r}_q(t)) = \underline{c\vec{R}(t)}$$

$$\vec{R}(t_r)\cdot\vec{u}(t_r) = \vec{R}(t_r)\cdot\left(c\hat{R}(t_r) - \vec{v}\right)$$

$$= cR(t_r) - \vec{R}(t_r)\cdot\vec{v} = c^2(t - t_r) - (\vec{r} - \vec{v}t_r)\cdot\vec{v}$$

$$= c^2 t - \vec{r}\cdot\vec{v} - (c^2 - v^2)\,t_r$$

We can find t_r from

$$c(t-t_r) = R(t_r) = |\vec{r} - \vec{v}t_r|$$

Squaring this gives $c^2(t-t_r)^2 = (\vec{r}-\vec{v}t_r)^2$

i.e.

$$c^2(t^2 - 2tt_r + t_r^2) = r^2 - 2\vec{r}\cdot\vec{v}t_r + v^2 t_r^2$$

$$\Rightarrow (c^2-v^2)t_r^2 + 2(\vec{r}\cdot\vec{v} - c^2 t)t_r + (c^2t^2 - r^2) = 0$$

This is a quadratic equation for t_r

$$\Rightarrow t_r = \frac{c^2t - \vec{r}\cdot\vec{v} \pm \sqrt{(c^2t-\vec{r}\cdot\vec{v})^2 - (c^2-v^2)(c^2t^2-r^2)}}{c^2-v^2}$$

Only one of these mathematical solutions is physical (since there is a unique physical solution for t_r). To identify it we consider the special case $\vec{v}=0$

$$\Rightarrow t_r = t \pm \frac{r}{c}$$ $\Rightarrow$ the minus sign gives the physical solution

Putting the physical solution for t_r into $\vec{R}(t_r)\cdot\vec{u}(t_r)$ gives

$$\vec{R}(t_r)\cdot\vec{u}(t_r) = c^2t - \vec{r}\cdot\vec{v} - (c^2-v^2)\frac{c^2t - \vec{r}\cdot\vec{v} - \sqrt{(c^2t-\vec{r}\cdot\vec{v})^2 - (c^2-v^2)(c^2t^2-r^2)}}{c^2-v^2}$$

$$= \sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 - (c^2-v^2)(c^2t^2-r^2)}$$

This expression can be rewritten in a more instructive form. To see this, define θ as the angle between $\vec{R}(t)$ and $\vec{v}$ and consider the expression

$$R^2(t)(c^2 - v^2\sin^2\theta) = R^2(t)(c^2-v^2) + R^2(t)v^2\cos^2\theta$$

$$= (\vec{r} - \vec{v}t)^2 (c^2 - v^2) + ((\vec{r} - \vec{v}t)\cdot\vec{v})^2$$

$$= (r^2 - 2\vec{r}\cdot\vec{v}t + v^2t^2)(c^2 - v^2)$$

$$+ (\vec{r}\cdot\vec{v})^2 - 2(\vec{r}\cdot\vec{v})v^2 t + v^4 t^2$$

$$= (\vec{r}\cdot\vec{v})^2 - 2\vec{r}\cdot\vec{v}\,c^2 t + (c^2 - v^2)r^2 + c^2v^2t^2$$

which can be seen to equal the expression inside the square root in $\vec{R}(t_r)\cdot\vec{u}(t_r)$

$$\Rightarrow \quad \vec{R}(t_r)\cdot\vec{u}(t_r) = cR(t)\sqrt{1 - \frac{v^2}{c^2}\sin^2\theta}$$

This gives

$$\underline{\underline{\vec{E}(\vec{r},t)}} = \frac{q}{4\pi\epsilon_0}\frac{c\vec{R}(t)\,(c^2 - v^2)}{c^3R^3(t)\left[1 - \frac{v^2}{c^2}\sin^2\theta\right]^{3/2}}$$

$$= \underline{\underline{\frac{q}{4\pi\epsilon_0}\frac{1 - v^2/c^2}{\left(1 - (v^2/c^2)\sin^2\theta\right)^{3/2}}\frac{\hat{R}(t)}{R^2(t)}}}$$

Thus the electric field $\vec{E}(\vec{r},t)$ at position $\vec{r}$ at time t has the direction of $\vec{R}(t) = \vec{r} - \vec{r}_q(t)$, ie. the difference vector between $\vec{r}$ and the particle's position $\vec{r}_q(t)$ at the <u>same</u> time t. This is rather interesting, given that fundamentally the field originates from the particle's position at the <u>retarded</u> time t_r.

The magnetic field $\vec{B}(\vec{r},t)$ is given by

$$\vec{B}(\vec{r},t) = \frac{1}{c}\hat{R}(t_r) \times \vec{E}(\vec{r},t)$$

Using that

$$\vec{R}(t_r) = \vec{r} - \vec{v}t_r = \vec{r} - \vec{v}t + (t - t_r)\vec{v}$$
$$= \vec{R}(t) + \frac{R(t_r)}{c}\vec{v}$$

we get $\hat{R}(t_r) = \frac{\vec{R}(t_r)}{R(t_r)} = \frac{\vec{R}(t)}{R(t_r)} + \frac{\vec{v}}{c}$

Thus

$$\vec{B}(\vec{r},t) = \frac{1}{c}\left(\frac{\vec{R}(t)}{R(t_r)} + \frac{\vec{v}}{c}\right) \times \vec{E}(\vec{r},t)$$

Since $\vec{E}(\vec{r},t) \parallel \vec{R}(t)$, $\vec{R}(t) \times \vec{E}(\vec{r},t) = 0$, which gives

$$\underline{\underline{\vec{B}(\vec{r},t) = \frac{1}{c^2}\vec{v} \times \vec{E}(\vec{r},t)}}$$

See Griffiths (Ex. 10.4) for figures illustrating $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$ for this case.

Problem 4

(a) The electric field at $(\vec{r}, t)$ due to particle 1 is just the electrostatic Coulomb field, since the particle is at rest:

$$\vec{E}_1(\vec{r}, t) = \frac{q_1}{4\pi\epsilon_0} \frac{\hat{r}}{r^2}$$

Particle 1 generates no magnetic field:

$$\vec{B}_1(\vec{r}, t) = 0$$

Note that these results for $\vec{E}_1$ and $\vec{B}_1$ can also be found by setting $\vec{v} = 0$ into the more general expressions for $\vec{E}(\vec{r}, t)$ and $\vec{B}(\vec{r}, t)$ from a particle moving at constant velocity $\vec{v}$.

The electromagnetic force from q_1 on q_2 at time t is therefore

$$\vec{F}_{12}(t) = q_2 \left(\vec{E}_1(\vec{r}_2(t), t) + \vec{v}_2 \times \underbrace{\vec{B}_1(\vec{r}_2(t), t)}_{=0} \right)$$

Using $\vec{r}_2(t) = \vec{v}_2 t$ with $\vec{v}_2 = v\hat{z}$ where $v > 0$ gives

$$\vec{F}_{12}(t) = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{\hat{r}_2(t)}{(\vec{r}_2(t))^2} = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{\hat{z}}{v^2 t^2}$$

(b) The electric field at $(\vec{r}, t)$ due to particle 2 is, since the particle moves at constant velocity,

$$\vec{E}_2(\vec{r}, t) = \frac{q_2}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{\left(1 - (v^2/c^2)\sin^2\theta_2\right)^{3/2}} \frac{\vec{r} - \vec{v}_2 t}{|\vec{r} - \vec{v}_2 t|^3}$$

where θ_2 is the angle between $\vec{v}_2$ and $\vec{r}-\vec{v}_2 t$.
The magnetic field due to particle 2 is

$$\vec{B}_2(\vec{r},t) = \frac{1}{c^2}\,\vec{v}_2 \times \vec{E}_2(\vec{r},t)$$

The electromagnetic force of q_2 on q_1 at time t is therefore

$$\vec{F}_{21}(t) = q_1\left(\vec{E}_2(\vec{r}_1(t),t) + \underbrace{\vec{v}_1}_{=0} \times \vec{B}_2(\vec{r}_1(t),t)\right)$$

Since $\vec{r}_1(t) = 0$, $\vec{r}_1(t) - \vec{v}_2 t = -\vec{v}_2 t$ and the angle θ becomes $\pi \Rightarrow \sin^2\theta = 0$

$$\Rightarrow \vec{F}_{21}(t) = \frac{q_1 q_2}{4\pi\epsilon_0}\frac{(-\hat{z})}{(\vec{v}_2 t)^2}(1 - v^2/c^2) = \underline{\underline{-\frac{q_1 q_2}{4\pi\epsilon_0}\frac{\hat{z}}{v^2 t^2}(1 - v^2/c^2)}}$$

The sum of the electromagnetic forces is

$$\vec{F}_{12}(t) + \vec{F}_{21}(t) = \frac{q_1 q_2}{4\pi\epsilon_0}\frac{\hat{z}}{c^2 t^2} \neq 0,$$

therefore Newton's 3rd law does not hold for the force pair consisting of $\vec{F}_{12}$ and $\vec{F}_{21}$.

(c) The momentum density of the fields is

$$\vec{g}_{EM} = \epsilon_0 \mu_0 \vec{S} = \frac{1}{c^2}\vec{S}$$

where $\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})$ is the Poynting vector

and $\vec{E}$ and $\vec{B}$ are the <u>total</u> fields, i.e.

$$\vec{E}(\vec{r},t) = \underbrace{\vec{E}_1(\vec{r},t)}_{\text{indep. of } t} + \vec{E}_2(\vec{r},t) = \vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r},t)$$

$$\vec{B}(\vec{r},t) = \vec{B}_1(\vec{r},t) + \vec{B}_2(\vec{r},t) = \vec{B}_2(\vec{r},t)$$

Thus $\vec{S}(\vec{r},t) = \frac{1}{\mu_0}\left(\vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r},t)\right) \times \vec{B}_2(\vec{r},t)$

$$= \frac{1}{\mu_0}\vec{E}_1(\vec{r}) \times \vec{B}_2(\vec{r},t) + \frac{1}{\mu_0}\vec{E}_2(\vec{r},t) \times \vec{B}_2(\vec{r},t)$$

The total momentum in the fields is

$$\vec{p}_{EM}(t) = \int d^3r\, \vec{g}_{EM}(\vec{r},t)$$

$$= \frac{1}{\mu_0 c^2}\int d^3r\, \vec{E}_1(\vec{r}) \times \vec{B}_2(\vec{r},t)$$

(the integration is over all space)

$$+ \frac{1}{\mu_0 c^2}\int d^3r\, \vec{E}_2(\vec{r},t) \times \vec{B}_2(\vec{r},t)$$

Consider the second integral here. While the integrand is time-dependent, the integration gives a result that is time-independent. To see this, we note that the integrand only involves the $\vec{E}$ and $\vec{B}$ fields of particle 2, which moves at constant velocity. Thus relative to the moving position of particle 2, the fields do not change. Thus if at each time t we pick the origin of space to be at the particle's position at that time (which we are free to do), the integral will be the same for all t and thus time-independent (note that the integral converges, because $\vec{E}_2 \times \vec{B}_2$ decays like $1/r^4$, so the r-part of the integral $\sim \int^{\infty} dr\, r^2 \cdot (1/r^4)$ for large r, which converges at the upper integration limit).

For reasons that will be discussed in (d), we don't bother calculating the time-independent contribution to $\vec{p}_{EM}(t)$, so we neglect the second integral from now on, and focus on the remaining first integral

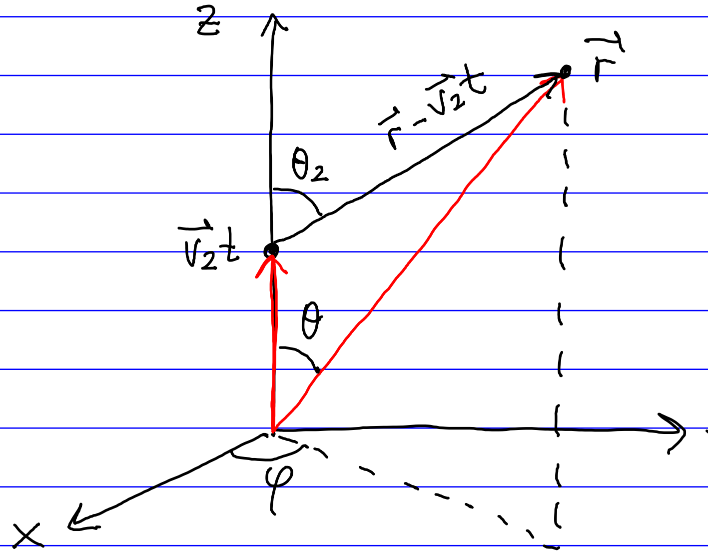
$$\frac{1}{\mu_0 c^2}\int d^3r \; \vec{E}_1(\vec{r}) \times \vec{B}_2(\vec{r},t)$$

$$\propto \int d^3r \; \frac{\hat{r}}{r^2} \times \left(\vec{v}_2 \times \frac{\vec{r}-\vec{v}_2 t}{|\vec{r}-\vec{v}_2 t|^3} \cdot \frac{1}{\left(1-(v^2/c^2)\sin^2\theta_2\right)^{3/2}} \right)$$

Using $\vec{v}_2 \times \vec{v}_2 = 0$ and $\hat{r}\times(\vec{v}_2\times\vec{r}) = \vec{v}_2(\hat{r}\cdot\vec{r}) - \vec{r}(\hat{r}\cdot\vec{v}_2)$ gives

$$(*) \quad \int d^3r \; \frac{1}{r^2}\,\frac{1}{|\vec{r}-\vec{v}_2 t|^3}\,\frac{1}{\left(1-(v^2/c^2)\sin^2\theta_2\right)^{3/2}}\left[\vec{v}_2(\hat{r}\cdot\vec{r}) - \vec{r}(\hat{r}\cdot\vec{v}_2)\right]$$

A figure is in order:



$\hat{r}\cdot\vec{r} = r$

$\hat{r}\cdot\vec{v}_2 = v\,\hat{z}\cdot\hat{r} = v\cos\theta$

$\vec{v}_2 = v\hat{z}$

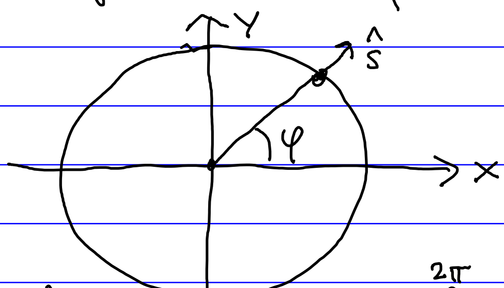
$\vec{r} = r\hat{r} = r(\sin\theta\,\hat{s} + \cos\theta\,\hat{z})$

$\Rightarrow \vec{v}_2(\hat{r}\cdot\vec{r}) - \vec{r}(\hat{r}\cdot\vec{v}_2)$

$= v r\hat{z} - v r\cos\theta(\cos\theta\,\hat{z} + \sin\theta\,\hat{s})$

$= \hat{z}\, v r\sin^2\theta - \hat{s}\, v r\cos\theta\sin\theta$

Let us consider the integral in spherical coordinates. Fix r and θ, and do the φ-integration. The only thing that depends on φ is the unit vector $\hat{s}$:



When φ goes from 0 to 2π, $\hat{s}$ rotates once $\Rightarrow \int_0^{2\pi} d\varphi\,\hat{s} = 0$

(more detailed: $\hat{s} = \hat{x}\cos\varphi + \hat{y}\sin\varphi \Rightarrow \int_0^{2\pi} d\varphi\,\hat{s} = \hat{x}\int_0^{2\pi} d\varphi\cos\varphi + \hat{y}\int_0^{2\pi} d\varphi\sin\varphi = \hat{x}\cdot 0 + \hat{y}\cdot 0 = 0$)

So the term $\propto \hat{s}$ will integrate to 0, while the term $\propto \hat{z}$ will just acquire a factor 2π from the φ-integration. For the remaining integrals over r and θ, we first have to rewrite everything in terms of these variables. We have (see figure)

$$|\vec{r}-\vec{v}_2 t| \sin\theta_2 = r\sin\theta \quad \Rightarrow \quad \sin^2\theta_2 = \frac{r^2}{|\vec{r}-\vec{v}_2 t|^2}\sin^2\theta$$

$$\Rightarrow \quad 1 - \frac{v^2}{c^2}\sin^2\theta_2 = 1 - \frac{v^2 r^2/c^2}{|\vec{r}-\vec{v}_2 t|^2}\sin^2\theta$$

$$= \frac{|\vec{r}-\vec{v}_2 t|^2 - (v^2 r^2/c^2)\sin^2\theta}{|\vec{r}-\vec{v}_2 t|^2}$$

$$\Rightarrow \quad |\vec{r}-\vec{v}_2 t|^3 \left(1 - \frac{v^2}{c^2}\sin^2\theta_2\right)^{3/2} = \left[|\vec{r}-\vec{v}_2 t|^2 - \frac{v^2 r^2}{c^2}\sin^2\theta\right]^{3/2}$$

Here, $|\vec{r}-\vec{v}_2 t|^2 = (\vec{r}-\vec{v}_2 t)^2 = r^2 - 2\vec{r}\cdot\vec{v}_2 t + \vec{v}_2^{\,2} t^2$

$$= r^2 - 2rvt\cos\theta + v^2 t^2$$

Thus the integral (*) on the previous page becomes

$$2\pi \int_{-1}^{1} d(\cos\theta) \int_0^\infty dr\, r^2 \cdot \frac{1}{r^2} \frac{\hat{z}\, v\, r \sin^2\theta}{\left[r^2\left(1-\frac{v^2}{c^2}\sin^2\theta\right) - 2rvt\cos\theta + v^2t^2\right]^{3/2}}$$

$$= \hat{z}\cdot 2\pi v \int_{-1}^{1} d(\cos\theta)\sin^2\theta \int_0^\infty dr \frac{r}{\left[r^2\left(1-\frac{v^2}{c^2}\sin^2\theta\right) - 2rvt\cos\theta + v^2t^2\right]^{3/2}}$$

The r-integral is on the form

$$\int dx \frac{x}{(ax^2+bx+c)^{3/2}} = \frac{2bx+4c}{(b^2-4ac)\sqrt{ax^2+bx+c}} + \text{konst.}$$

with $a = 1 - \frac{v^2}{c^2}\sin^2\theta$, $b = -2vt\cos\theta$, $c = v^2t^2$

$$\Rightarrow \hat{z}\cdot 2\pi v \int_{-1}^{1} d(\cos\theta)\ \sin^2\theta \cdot \frac{1}{b^2-4ac}\cdot\left[\underbrace{\frac{2b}{\sqrt{a}}}_{\text{from } r=\infty} - \underbrace{\frac{4c}{\sqrt{c}}}_{\text{from } r=0}\right]$$

$$= \hat{z}\cdot 2\pi v \int_{-1}^{1} d(\cos\theta)\ \sin^2\theta\ \frac{\frac{-4vt\cos\theta}{\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}} - 4vt}{4v^2t^2\cos^2\theta - 4\left(1-\frac{v^2}{c^2}\sin^2\theta\right)v^2t^2}$$

$$= \hat{z}\ \frac{2\pi}{t}\ \frac{1}{1-v^2/c^2}\int_{-1}^{1} d(\cos\theta)\left[1 + \frac{\cos\theta}{\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}}\right]$$

The last integral vanishes because the integrand is odd in $x=\cos\theta$ $(\sin^2\theta = 1-\cos^2\theta = 1-x^2)$

$$\Rightarrow \hat{z}\ \frac{4\pi}{t}\ \frac{1}{1-v^2/c^2}$$

Inserting the constant prefactors, we get

$$\vec{p}_{EM}(t) = \frac{1}{\mu_0 c^2}\ \frac{q_1}{4\pi\epsilon_0}\ \frac{1}{c^2}\ \frac{q_2}{\cancel{4\pi}\epsilon_0}\cancel{\left(1-v^2/c^2\right)}\ \frac{\cancel{4\pi}}{t}\ \cancel{\frac{1}{1-v^2/c^2}}\ \hat{z} + \text{const.}$$

$$= \boxed{\frac{\mu_0 q_1 q_2}{4\pi t}\ \hat{z} + \text{const.}}$$

(where "const." is the time-independent contribution we didn't bother calculating).

(d) We see that

$$-\frac{d\vec{p}_{EM}}{dt} = \frac{\mu_0 q_1 q_2}{4\pi t^2}\ \hat{z} = \vec{F}_{12}(t) + \vec{F}_{21}(t) \qquad (**)$$

This result can be interpreted in terms of the equation

$$\vec{F} = \oint \overleftrightarrow{T}\cdot d\vec{a} - \frac{d\vec{p}_{EM}}{dt} \qquad (***)$$

that we derived earlier (see Eq. (8.22) in Griffiths). Here $\vec{F}$ is the total electromagnetic force on the charges in the volume Ω (the force due to the fields generated by the charges) and $\oint \overleftrightarrow{T} \cdot d\vec{a}$ is the surface integral of Maxwell's stress tensor over the boundary of Ω. When Ω = all space this surface integral vanishes (recall T scales like E^2 and B^2 and thus decays fast enough for the surface integral to vanish at infinity), so (***) becomes

$$\vec{F} = - \frac{d\vec{p}_{EM}}{dt}$$

Using also that $\vec{F} = \vec{F}_{12} + \vec{F}_{21}$ in our system gives exactly our result (**) on the previous page.

A remark: Note that $\vec{F}$ is here the total electromagnetic force due to the fields generated by the charges. In this problem there are also other, external forces involved. That is clear from the fact that both charges move at <u>constant</u> velocity ($\vec{v}_1 = 0$ and $\vec{v}_2 = v\hat{z}$) so the net force on each particle is zero. Therefore there must also be external forces acting on each charge which balance the force $\vec{F}_{21}$ on particle 1 and the force $\vec{F}_{12}$ on particle 2. But these external forces are not part of the "accounting" expressed by the equation (***).