Vector analysis

Abstract

These notes present some background material on vector analysis. Except for the material related to proving vector identities (including Einstein's summation convention and the Levi-Civita symbol), the topics are discussed in more detail in Griffiths.

Contents

1 Scalars and vectors. Fields. Coordinate systems **1**

2 The ∇ operator **2**
2.1 The gradient, divergence, curl, and Laplacian 3

3 Vector identities **3**
3.1 Proving the vector identities 5

4 Some integral theorems **8**
4.1 The divergence theorem 9
4.2 Stokes' theorem 9

5 Some results involving the Dirac delta function **9**
5.1 The Dirac delta function 9
5.2 Some useful results 10

1 Scalars and vectors. Fields. Coordinate systems

Scalars and vectors are two important concepts in this course. A scalar f is a quantity that is a real number (although not all real-numbered quantites are scalars; see below), while a vector $\boldsymbol{v}$ is a quantity that has both a magnitude and a direction (usually thought of as an arrow with a certain length, pointing in a certain direction).

Quantities that can be defined at each point $\boldsymbol{r}$ in space (and that can generally vary with $\boldsymbol{r}$) are commonly called fields. We will encounter both scalar fields $f(\boldsymbol{r})$ and vector fields $\boldsymbol{v}(\boldsymbol{r})$ in this course. In general these may depend on time as well: $f(\boldsymbol{r}, t)$ and $\boldsymbol{v}(\boldsymbol{r}, t)$. An example of a scalar field is the electric charge density ρ. Examples of vector fields are the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$. Note that we will often not write the dependence on $\boldsymbol{r}$ and t explicitly, thus writing just f and $\boldsymbol{v}$ instead of $f(\boldsymbol{r}, t)$ and $\boldsymbol{v}(\boldsymbol{r}, t)$.

Although in principle vectors can be analyzed independently of any coordinate system, it is in practice often very useful to represent a vector in terms of its components with respect to a particular coordinate system. Focusing here on 3-dimensional vectors, we will use coordinate systems with basis vectors $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, $\hat{\boldsymbol{e}}_3$ which have unit length, are mutually orthogonal and span a right-handed coordinate system. In other words,

$$\begin{array}{llll} \hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_1 = 1, & \hat{\boldsymbol{e}}_2 \cdot \hat{\boldsymbol{e}}_2 = 1, & \hat{\boldsymbol{e}}_3 \cdot \hat{\boldsymbol{e}}_3 = 1, & \text{(unit length)} \\ \hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_2 = 0, & \hat{\boldsymbol{e}}_2 \cdot \hat{\boldsymbol{e}}_3 = 0, & \hat{\boldsymbol{e}}_3 \cdot \hat{\boldsymbol{e}}_1 = 0, & \text{(orthogonality)} \\ \hat{\boldsymbol{e}}_1 \times \hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{e}}_3, & \hat{\boldsymbol{e}}_2 \times \hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{e}}_1, & \hat{\boldsymbol{e}}_3 \times \hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{e}}_2. & \text{(righthandedness)} \end{array} \tag{1}$$

An arbitrary vector $\boldsymbol{v}$ can then be expressed as

$$\boldsymbol{v} = \sum_{i=1}^{3} v_i \hat{\boldsymbol{e}}_i \tag{2}$$

where the components v_i in this basis are given by $v_i = \hat{\boldsymbol{e}}_i \cdot \boldsymbol{v}$, i.e. the projections of $\boldsymbol{v}$ along the basis vectors. Thus if we switch to a different coordinate system (e.g. one with its axes rotated with respect to those of the original one), the vector's components will change. Also note that a scalar is more precisely defined as a quantity that is not affected by a change of coordinates. Thus although vector components are real numbers, they are not scalars.

In this course we will use the cartesian, spherical and cylindrical coordinate systems (see Table 1 and Fig. 1).[1] One should pick the coordinate system that is most convenient for the particular problem one wishes to solve. For generic problems the cartesian coordinate system is often the most convenient one. The spherical and cylindrical coordinate systems can be more convenient if the problem under consideration has spherical or cylindrical symmetry, respectively.[2]

Coordinate system	Coordinates x_1, x_2, x_3	Basis vectors $\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3$
Cartesian	x, y, z	$\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$
Spherical	r, θ, ϕ	$\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$
Cylindrical	s, ϕ, z	$\hat{\boldsymbol{s}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{z}}$

Table 1: Coordinates and basis vectors for the coordinate systems (see also Fig. 1).

2 The $\boldsymbol{\nabla}$ operator

As we will analyze quantities varying in space, the vector operator $\boldsymbol{\nabla}$ (called the nabla, del or gradient operator) will play a central role. In the cartesian coordinate system $\boldsymbol{\nabla}$ is given by

$$\boldsymbol{\nabla} = \hat{\boldsymbol{x}} \frac{\partial}{\partial x} + \hat{\boldsymbol{y}} \frac{\partial}{\partial y} + \hat{\boldsymbol{z}} \frac{\partial}{\partial z}. \tag{3}$$

[1] For the cylindrical coordinate system, the radial coordinate in the xy plane is commonly denoted by ρ. However, we will use s instead, as the letter ρ will be reserved for the charge density.

[2] Note that, unlike the basis vectors $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$ which are constant, the basis vectors $\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{s}}$ vary from point to point in space. Thus when working in spherical or cylindrical coordinates, one must keep in mind that spatial derivatives of vector functions will get contributions not just from the vector components, but also from the basis vectors.

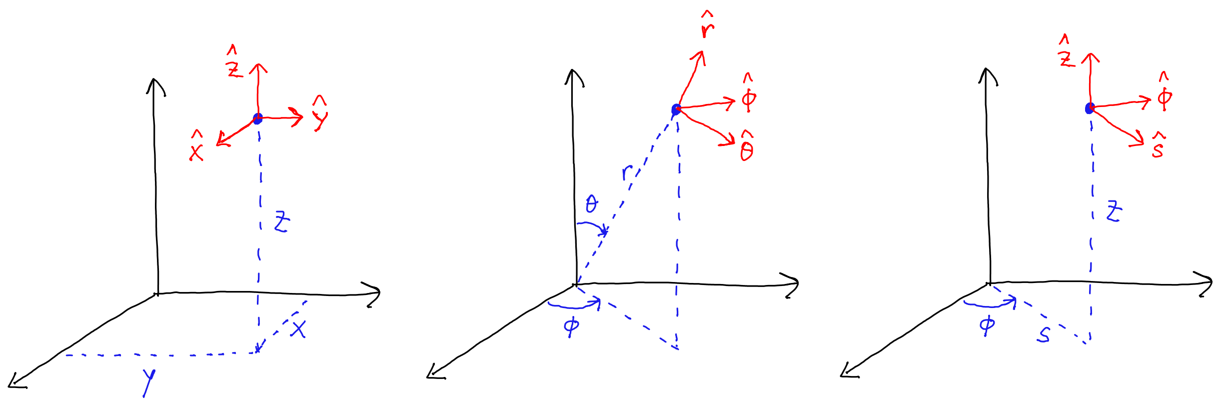

Figure 1: Coordinates of, and basis vectors at, an arbitrary point in space: Cartesian (left), spherical (middle), cylindrical (right).

2.1 The gradient, divergence, curl, and Laplacian

The most important quantities involving $\boldsymbol{\nabla}$('s) acting in various ways on scalar functions $f(\boldsymbol{r})$ or vector functions $\boldsymbol{v}(\boldsymbol{r})$ include the gradient, divergence, curl and Laplacian. Some of their basic properties are listed in Table 2. See Fig. 2 for expressions for these quantities in the various coordinate systems. The expressions take their simplest form in the cartesian coordinate system, where they follow quite straightforwardly from using the expression (3) and the standard definitions of the scalar (dot) and vector (cross) product. For a derivation of the expressions applicable to all three coordinate systems, see e.g. Appendix A in Griffiths. Alternatively, expressions can be converted from one coordinate system to another with help from the chain rule.

Quantity	Alternative notation	Name	Type of mapping
$\boldsymbol{\nabla} f$	grad f	gradient	scalar $\to$ vector
$\boldsymbol{\nabla}\cdot\boldsymbol{v}$	div $\boldsymbol{v}$	divergence	vector $\to$ scalar
$\boldsymbol{\nabla}\times\boldsymbol{v}$	curl $\boldsymbol{v}$	curl	vector $\to$ vector
$\nabla^2 f$ ($\nabla^2\boldsymbol{v}$)		Laplacian	scalar $\to$ scalar (vector $\to$ vector)

Table 2: The most important quantities involving $\boldsymbol{\nabla}$('s) acting on scalar functions $f(\boldsymbol{r})$ and vector functions $\boldsymbol{v}(\boldsymbol{r})$. The Laplacian is the divergence of the gradient: $\nabla^2 \equiv \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$.

3 Vector identities

A list of vector identities[3] is given in Fig. 3. Here f and g are arbitrary scalar fields (including constant scalars as a special case), and $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are arbitrary vector fields (including constant vectors as a special case). These quantities may depend on additional variables as well, e.g. time.

[3]Although some of the identities are for dot products and therefore scalars, not vectors, all identities *involve* vectors, so for simplicity we refer to all of them collectively as vector identities.

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

$$\text{Gradient:} \qquad \nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

$$\text{Divergence:} \qquad \nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\text{Curl:} \qquad \nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

$$\text{Laplacian:} \qquad \nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

$$\text{Gradient:} \qquad \nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

$$\text{Divergence:} \qquad \nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

$$\text{Curl:} \qquad \nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

$$\text{Laplacian:} \qquad \nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

$$\text{Gradient:} \qquad \nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

$$\text{Divergence:} \qquad \nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

$$\text{Curl:} \qquad \nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

$$\text{Laplacian:} \qquad \nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

Figure 2: Expressions for the gradient, divergence, curl, and Laplacian in the cartesian, spherical, and cylindrical coordinate systems (copy from Griffiths).

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

Figure 3: A list of vector identities (copy from Griffiths).

Two of the most important identities are (9) and (10), which say, respectively, that a curl has zero divergence and a gradient has zero curl.

3.1 Proving the vector identities

Since we will use many of these identities many times in this course, we should know how to prove them. For this purpose we will now introduce some new quantities and notation that will be very helpful in simplifying such proofs.

- We introduce the **Kronecker delta symbol** δ_{ij}, defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{4}$$

  Thus we can e.g. write the 6 equations in the first and second line of (1) much more simply as the single expression

$$\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j = \delta_{ij}. \tag{5}$$

We will often encounter sums involving the Kronecker delta symbol. These are very easy to evaluate. For example,

$$\sum_j \delta_{ij} A_j = A_i. \tag{6}$$

- In cartesian coordinates ($x_1 = x$, $x_2 = y$, $x_3 = z$) we define

$$\partial_i \equiv \frac{\partial}{\partial x_i}. \tag{7}$$

Thus we can e.g. write $\frac{\partial f}{\partial x_i}$ and $\frac{\partial^2 f}{\partial x_i \partial x_j}$ more simply as $\partial_i f$ and $\partial_i \partial_j f$, respectively.

- The spatial indices like i and j in the expressions above take values 1, 2 or 3. We now introduce the following rule: If a spatial index appears exactly twice in a term, that index should be summed from 1 to 3. This rule is called **Einstein's summation convention** (ESC). Here are some examples (Cartesian coordinates are used)

$$\boldsymbol{v} = \sum_i v_i \hat{\boldsymbol{e}}_i \stackrel{\text{ESC}}{=} v_i \hat{\boldsymbol{e}}_i, \tag{8}$$

$$\boldsymbol{\nabla} f = \sum_i \hat{\boldsymbol{e}}_i \partial_i f \stackrel{\text{ESC}}{=} \hat{\boldsymbol{e}}_i \partial_i f, \tag{9}$$

$$\boldsymbol{v} \cdot \boldsymbol{w} = \sum_i v_i w_i \stackrel{\text{ESC}}{=} v_i w_i, \tag{10}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = \sum_i \partial_i v_i \stackrel{\text{ESC}}{=} \partial_i v_i, \tag{11}$$

$$\sum_j \sum_l \sum_m \delta_{il} \delta_{jm} A_j B_l C_m \stackrel{\text{ESC}}{=} \delta_{il} \delta_{jm} A_j B_l C_m. \tag{12}$$

We see that ESC shortens the expressions by eliminating summation signs for indices appearing twice. Such indices are also called repeated indices or dummy indices. Indices that only appear once are called free indices. In contrast to free indices, dummy indices can be renamed at will. For example, $\partial_i v_i = \partial_j v_j$. The only constraint is that the new dummy index name (here j) cannot already be used elsewhere in the term. Terms that are added together (including all terms appearing in an equality) must have exactly the same free indices.

If, when using the ESC, one wants to write an expression in which repeated indices should *not* be summed over (this usually happens quite rarely), one has to indicate that by including an explicit comment like "no sum". For example, the first line in (1) could be written

$$\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_i = 1 \quad \text{(no sum over } i\text{)} \tag{13}$$

- The **Levi-Civita symbol** (sometimes called the totally antisymmetric symbol) ϵ_{ijk} (where i, j and k are all spatial indices taking values 1, 2, or 3) can be defined by the following two requirements:

(i) ϵ_{ijk} changes sign when any two indices are permuted (14)

(ii) $\epsilon_{123} \equiv 1$. (15)

For example, it then follows that $\epsilon_{213} = -\epsilon_{123} = -1$, $\epsilon_{312} = -\epsilon_{213} = -(-1) = 1$, and $\epsilon_{112} = -\epsilon_{112} = 0$. In the last example we permuted the two indices equal to 1, giving a minus sign. This shows that ϵ_{112} is equal to minus itself, and therefore it must be 0. More generally, one can show that $\epsilon_{ijk} = 0$ if two or more indices are the same. Working out all components, one finds that

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123, 231, 312, \\ -1 & \text{if } ijk = 132, 321, 213, \\ 0 & \text{otherwise (two or more indices the same).} \end{cases} \tag{16}$$

Note that 123, 231, and 312 are cyclic permutations of 123, while 132, 321 and 213 are anticyclic permutations. Thus $\epsilon_{ijk} = 1\,(-1)$ if ijk is a cyclic (anticyclic) permutation of 123; otherwise, $\epsilon_{ijk} = 0$. The Levi-Civita symbol is unchanged by cyclic permutations of the indices: $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$. It can also be shown that

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}, \tag{17}$$

which is a sum that will turn out to be very useful.

We are now ready to start proving vector identities. In these proofs we will always use the cartesian coordinate system, expressing everything in terms of cartesian components (for example, in the following, A_i is the ith cartesian component of $\boldsymbol{A}$). We will also use the Einstein summation convention. The Levi-Civita symbol makes its entrance via the fact that (you should verify it)

$$\text{if } \boldsymbol{A} = \boldsymbol{B} \times \boldsymbol{C} \quad \text{then} \quad A_i = \epsilon_{ijk} B_j C_k. \tag{18}$$

The same form holds also if the vector function $\boldsymbol{B}$ is replaced by the vector operator $\boldsymbol{\nabla}$, i.e.

$$(\boldsymbol{\nabla} \times \boldsymbol{C})_i = \epsilon_{ijk} \partial_j C_k. \tag{19}$$

As a warm-up, let us first verify the property $\boldsymbol{A} \times \boldsymbol{B} = -\boldsymbol{B} \times \boldsymbol{A}$. As this expresses an equality of two vectors, we can prove it by proving that it holds for an arbitrary component $i = x, y, z$ of the vectors. Thus consider

$$\begin{aligned} (\boldsymbol{A} \times \boldsymbol{B})_i &= \epsilon_{ijk} A_j B_k = \epsilon_{ijk} B_k A_j \\ &= -\epsilon_{ikj} B_k A_j \qquad \text{(permuted indices } j, k \text{ in LC symbol)} \\ &= -(\boldsymbol{B} \times \boldsymbol{A})_i. \end{aligned} \tag{20}$$

Next, we prove the triple product identity (1) in Fig. 3.

$$\begin{aligned} \boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) &= A_i (\boldsymbol{B} \times \boldsymbol{C})_i \\ &= A_i \epsilon_{ijk} B_j C_k = B_j \epsilon_{ijk} C_k A_i \\ &= B_j \epsilon_{jki} C_k A_i \qquad \text{(cyclically permuted the indices in LC symbol)} \\ &= B_j (\boldsymbol{C} \times \boldsymbol{A})_j \\ &= \boldsymbol{B} \cdot (\boldsymbol{C} \times \boldsymbol{A}). \end{aligned} \tag{21}$$

Thus we have proven the first equality in (1). It says that this triple product is invariant under a cyclic permutation of the vectors. The second equality in (1) follows automatically from doing another cyclic permutation.

Next, we prove the triple product identity (2) in Fig. 3:

$$\begin{aligned}
[\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})]_i &= \epsilon_{ijk} A_j (\boldsymbol{B} \times \boldsymbol{C})_k \\
&= \epsilon_{ijk} A_j \epsilon_{klm} B_l C_m = \epsilon_{ijk} \epsilon_{klm} A_j B_l C_m \\
&= \epsilon_{kij} \epsilon_{klm} A_j B_l C_m \qquad \text{(cyclically permuted indices in first LC symbol))} \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) A_j B_l C_m \qquad \text{(did sum over } k \text{ using (17))} \\
&= \delta_{il}\delta_{jm} A_j B_l C_m - \delta_{im}\delta_{jl} A_j B_l C_m \\
&= A_j B_i C_j - A_j B_j C_i \qquad \text{(did sums over } m \text{ and } l\text{)} \\
&= B_i \boldsymbol{A} \cdot \boldsymbol{C} - C_i \boldsymbol{A} \cdot \boldsymbol{B}. \qquad (22)
\end{aligned}$$

As our final example, we prove identity (10) in Fig. 3, which says that gradients have zero curl:

$$\begin{aligned}
[\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f)]_i &= \epsilon_{ijk} \partial_j (\boldsymbol{\nabla} f)_k = \epsilon_{ijk} \partial_j \partial_k f \\
&= -\epsilon_{ikj} \partial_j \partial_k f \qquad \text{(permuted indices } j, k \text{ in LC symbol)} \\
&= -\epsilon_{ikj} \partial_k \partial_j f \qquad \text{(switched the order of the partial derivatives)} \\
&= -\epsilon_{ijk} \partial_j \partial_k f \qquad \text{(switched the names of the dummy indices } j, k\text{).} \quad (23)
\end{aligned}$$

This shows that the component is equal to minus itself and hence equals zero.

The proof of some of the other identities will be left as tutorial problems.

4 Some integral theorems

The divergence theorem and Stokes' theorem[4] are two very useful integral theorems involving vector functions. We state them here without proof.

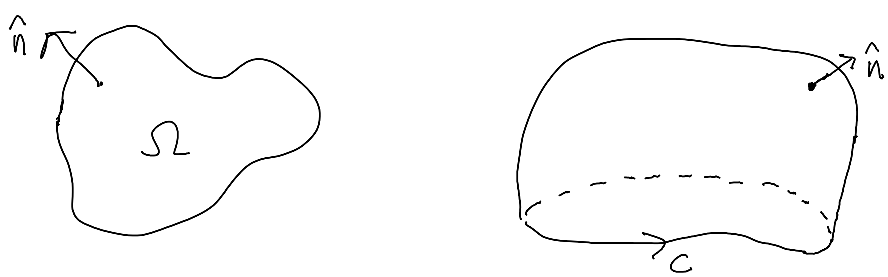

Figure 4: Left: A volume enclosed by a surface. Right: An open surface bounded by a loop.

[4]Stokes' theorem is called the curl theorem by Griffiths.

4.1 The divergence theorem

Consider a volume Ω enclosed by a surface a (see Fig. 4 (left)).[5] At each point on the surface there is a unit vector $\hat{\boldsymbol{n}}$ that points perpendicularly to the surface, in the direction out of the surface. For an infinitesimal surface element with area da, define $d\boldsymbol{a} = \hat{\boldsymbol{n}}\, da$. Let $\boldsymbol{v}(\boldsymbol{r})$ be a vector function (vector field). The divergence theorem states that

$$\int_{\Omega} d^3r\, \boldsymbol{\nabla} \cdot \boldsymbol{v} = \int_a d\boldsymbol{a} \cdot \boldsymbol{v}. \tag{24}$$

In words: The volume integral of the divergence of $\boldsymbol{v}$ equals the flux of $\boldsymbol{v}$ out of the closed surface bounding the volume.

4.2 Stokes' theorem

Consider an open surface a (see Fig. 4 (right)). Define a normal unit vector field $\hat{\boldsymbol{n}}$ as above (except that since the surface here is not closed, the definition of the direction "out of" the surface is ambiguous, so we must just pick one of the two possibilities). As the surface is open, it is bounded by a closed curve C. Let C consist of infinitesimal line elements $d\boldsymbol{\ell}$ whose direction is determined by a right-hand rule: Curling one's right hand's fingers around C in the direction of $d\boldsymbol{\ell}$, the thumb points in the direction of $\hat{\boldsymbol{n}}$. Stokes' theorem states that

$$\int_a d\boldsymbol{a} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}) = \oint_C d\boldsymbol{\ell} \cdot \boldsymbol{v}. \tag{25}$$

In words: The flux of curl $\boldsymbol{v}$ through the surface equals the line integral of $\boldsymbol{v}$ around the loop bounding the surface.

5 Some results involving the Dirac delta function

5.1 The Dirac delta function

The most basic properties of the Dirac delta function in one and three dimensions are summarized in Table 3.[6]

Dimension	Definition	Normalization
$d = 1$	$\int_{-\infty}^{\infty} dx\, \delta(x - x')f(x) = f(x')$	$\int_{-\infty}^{\infty} dx\, \delta(x - x') = 1$
$d = 3$	$\int_{\text{all space}} d^3r\, \delta(\boldsymbol{r} - \boldsymbol{r}')f(\boldsymbol{r}) = f(\boldsymbol{r}')$	$\int_{\text{all space}} d^3r\, \delta(\boldsymbol{r} - \boldsymbol{r}') = 1$

Table 3: Basic properties of the Dirac delta function in 1 and 3 dimensions.

Some comments:

[5]It is common to use S to denote surfaces and surface areas. However, we use a instead because in this course S will be used for the so-called Poynting vector.

[6]Only a very brief discussion is given here, as it is assumed that you have had some prior exposure to this function in other mathematics and/or physics courses.

- The normalization follows from the definition by choosing the function f equal to 1.
- For both the 1D and 3D integrals the integration region given is all space. However, it is not necessary to integrate over all space to get the results on the rhs. It is sufficient to integrate over a region that contains the point where the delta function is nonzero. If the integration region does not contain this point, the integral evalutes to 0.
- The integrals show that the Dirac delta function has dimensions of inverse length in 1D and inverse volume in 3D.
- The 3D Dirac delta function can be expressed in terms of 1D Dirac delta functions as[7]

$$\delta(\boldsymbol{r} - \boldsymbol{r}') = \delta(x - x')\delta(y - y')\delta(z - z'). \tag{26}$$

- The Dirac delta function $\delta(x - x')$ is not an ordinary function, but is instead what is called a generalized function in mathematics. For our purposes it is however sufficiently accurate to think of it as a limit of an ordinary function. This function is zero everywhere except for a very high and very thin peak centered at $x = x'$, such that the area under the function equals 1. This function becomes $\delta(x-x')$ in the limit that the peak height goes to infinity and the peak width goes to zero in such a way that the area under the function remains fixed to 1. (Various explicit representations of such ordinary functions are also used, but we will not need to go into that here.)

5.2 Some useful results

Define $\boldsymbol{R} \equiv \boldsymbol{r} - \boldsymbol{r}'$, $R \equiv |\boldsymbol{R}|$, and $\hat{\boldsymbol{R}} \equiv \boldsymbol{R}/R$. Also note that $\boldsymbol{\nabla}$ refers (as usual) to differentiation with respect to $\boldsymbol{r}$ (not $\boldsymbol{r}'$ or $\boldsymbol{R}$). The following expressions hold:

$$\boldsymbol{\nabla} \cdot \frac{\hat{\boldsymbol{R}}}{R^2} = 4\pi\delta(\boldsymbol{R}), \tag{27}$$

$$\boldsymbol{\nabla}\frac{1}{R} = -\frac{\hat{\boldsymbol{R}}}{R^2}, \tag{28}$$

$$\nabla^2\frac{1}{R} = -4\pi\delta(\boldsymbol{R}). \tag{29}$$

We will see later that these results are useful in electrostatics. To prove them, we will first assume that $\boldsymbol{r}' = \boldsymbol{0}$, i.e. $\boldsymbol{R} = \boldsymbol{r}$, and later generalize to $\boldsymbol{r}' \neq \boldsymbol{0}$.

Thus consider $\nabla \cdot (\hat{\boldsymbol{r}}/r^2)$. This is the divergence of a vector $\boldsymbol{v}$ expressed in spherical coordinates, with components $v_r = 1/r^2$ and $v_\theta = v_\phi = 0$. Using the appropriate expression in Fig. 2 we get

$$\boldsymbol{\nabla} \cdot \frac{\hat{\boldsymbol{r}}}{r^2} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{1}{r^2}) = \frac{1}{r^2}\frac{\partial}{\partial r}1 = \frac{1}{r^2}\cdot 0 = \begin{cases} 0 & \text{if } r \neq 0, \\ \frac{0}{0} & \text{if } r = 0. \end{cases} \tag{30}$$

[7] As the 3D and 1D delta functions are obviously different functions, one should strictly speaking use different names for them. For example, one could let δ denote the 1D function and $\delta^{(3)}$ denote the 3D function. However, we will instead use the same name δ for both, letting its argument tell whether we mean the 1D or 3D function (this "abuse of notation" is quite common in physics).

Thus for $r \neq 0$ the divergence vanishes, while for $r = 0$ this direct calculation fails to give an answer, as 0/0 is ill-defined. Let us therefore try a more indirect approach: we integrate the divergence over a volume Ω that is a sphere of radius $\bar{r}$ centered at the origin, and then use the divergence theorem to rewrite this as an integral over the spherical surface a:

$$\int_\Omega d^3r\, \nabla \cdot \boldsymbol{v} = \int_a d\boldsymbol{a} \cdot \boldsymbol{v} = \int_a \hat{\boldsymbol{r}} da \cdot \frac{\hat{\boldsymbol{r}}}{\bar{r}^2} = \frac{1}{\bar{r}^2} \int da = \frac{1}{\bar{r}^2} 4\pi\bar{r}^2 = 4\pi. \tag{31}$$

The only way to get this *nonzero* result for the volume integral, given that the integrand vanishes for all $r \neq 0$, is that the integrand must be proportional to a Dirac delta function centered at $r = 0$, with weight 4π:

$$\boldsymbol{\nabla} \cdot \frac{\hat{\boldsymbol{r}}}{r^2} = 4\pi\delta(\boldsymbol{r}). \tag{32}$$

Indeed, a direct volume integration of this result (trivial due to the delta function form) gives $\int_\Omega d^3r \nabla \cdot (\hat{\boldsymbol{r}}/r^2) = 4\pi$, in agreement with (31).

Next consider $\boldsymbol{\nabla}(1/r)$. This is of the form ∇f in spherical coordinates, with $f = 1/r$ which thus has no θ or ϕ dependence. Thus the appropriate expression in Fig. 2 reduces to

$$\boldsymbol{\nabla}\frac{1}{r} = \hat{\boldsymbol{r}}\frac{\partial}{\partial r}\frac{1}{r} = -\frac{\hat{\boldsymbol{r}}}{r^2}. \tag{33}$$

Finally, using (33) and (32) gives

$$\nabla^2\frac{1}{r} = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\frac{1}{r} = \boldsymbol{\nabla} \cdot \left(-\frac{\hat{\boldsymbol{r}}}{r^2}\right) = -4\pi\delta(\boldsymbol{r}). \tag{34}$$

Next, we must generalize the derivations to $\boldsymbol{r}' \neq 0$. To do this, we give a visual/geometric argument (of course, the same conclusions could be arrived at by explicit calculations). We note that the only difference between the functions $\hat{\boldsymbol{R}}/R^2$ and $1/R$ appearing on the left-hand sides of (27)-(29) and the corresponding functions $\hat{\boldsymbol{r}}/r^2$ and $1/r$ appearing on the left-hand sides of (32)-(34) is that the former are centered at $\boldsymbol{r} = \boldsymbol{r}'$ while the latter are centered at $\boldsymbol{r} = \boldsymbol{0}$. In other words: Measured with respect to their respective centers, their shapes are identical. As the ∇'s probe the spatial structure of these functions, the right-hand sides (rhs's) of the equations are related by the same shift. Thus to get the rhs's of (27)-(29) we just need to do the shift $\boldsymbol{r} \to \boldsymbol{r} - \boldsymbol{r}' = \boldsymbol{R}$ in the rhs's of (32)-(34).