Solution to week 3 exercises

Exercise 1.5.3

We are asked to find the extremal path between two arbitrary points (θ_0, ϕ_0) and (θ_1, θ_1) on the sphere. In spherical coordinates, the arc length is given by

$$\begin{aligned} ds &= \sqrt{d\theta^2 + \sin^2\theta \, d\phi^2} \\ &= \sqrt{1 + \sin^2\theta (\phi')^2} \, d\theta \ , \end{aligned} \tag{1}$$

where we have parameterized ϕ as a function of θ and $\phi' = \frac{d\phi}{d\theta}$. coordinate. The Euler-Lagrange equation reads

$$\frac{d}{d\theta}\frac{\partial G}{\partial \phi'} = \frac{\partial G}{\partial \phi} \ . \tag{2}$$

We note that ϕ is a cyclic and the right-hand side of Eq. (2) thus vanishes. Integration of the left-hand side yields

$$\frac{\sin^2\theta\phi'}{\sqrt{1 + \sin^2\theta(\phi')^2}} = c \ , \tag{3}$$

where c is some constant. Since the shape of the curve cannot depend on where we place the north pole on the sphere, we may choose to place the north pole at the starting point, in other words we choose our coordinates such that

$$\theta_0 = 0$$

This yields

$$c = 0$$

So for $\theta \neq 0$, we thus see

$$\phi' = 0$$

and thus

$$\phi = k$$

where k is some constant. We have thus obtained the result that the great circle is the geodesic of a sphere.

Exercise 2.5.4

The transformations that relate the coordinate differences between two events in space-time are

$$\Delta x' = \gamma(\Delta x - v\Delta t) \ , \quad (4)$$
$$\Delta t' = \gamma(\Delta t - v\Delta x/c^2) \ , \quad (5)$$

If the two events take place at the same time in S', the left-hand side of Eq. (5) vanishes and we obtain

$$\Delta t - v\Delta x/c^2 = 0 \ . \quad (6)$$

This corresponds to a boost velocity of

$$v = c^2 \frac{\Delta t}{\Delta x} \ . \quad (7)$$

Since the interval is spacelike, we have

$$c^2\Delta t^2 - \Delta x^2 < 0 \ . \quad (8)$$

This implies that the boost velocity satisfies $v < c$ and so this defines an admissible inertial frame. If two events take place at the same point in space, the left-hand side of Eq. (4) vanishes and we obtain

$$(\Delta x - v\Delta t) = 0 \ . \quad (9)$$

This corresponds to a boost velocity of

$$v = \frac{\Delta x}{\Delta t} \ . \quad (10)$$

Since the interval is timelike, we have

$$c^2\Delta t^2 - \Delta x^2 > 0 \ . \quad (11)$$

This implies that the boost velocity satisfies $v < c$ and so this defines an admissible inertial frame.

Exercise 2.5.6

Define the parameters θ and θ' by $\tanh\theta = v/c$ and $\tanh\theta' = v'/c$ (these parameters are often referred to as "hyperbolic angles" or "rapidities"). Then

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix} , \quad (12)$$
$$\begin{pmatrix} x'' \\ ct'' \end{pmatrix} = \begin{pmatrix} \cosh\theta' & -\sinh\theta' \\ -\sinh\theta' & \cosh\theta' \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix} . \quad (13)$$

Thus

$$\begin{aligned}
\begin{pmatrix} x'' \\ ct'' \end{pmatrix} &= \begin{pmatrix} \cosh\theta' & -\sinh\theta' \\ -\sinh\theta' & \cosh\theta' \end{pmatrix} \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix} \\
&= \begin{pmatrix} \cosh\theta\cosh\theta' + \sinh\theta\sinh\theta' & -\sinh\theta\cosh\theta' - \cosh\theta\sinh\theta' \\ -\sinh\theta\cosh\theta' - \cosh\theta\sinh\theta' & \cosh\theta\cosh\theta' + \sinh\theta\sinh\theta' \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix} \\
&= \begin{pmatrix} \cosh(\theta+\theta') & -\sinh(\theta+\theta') \\ -\sinh(\theta+\theta') & \cosh(\theta+\theta') \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}. \qquad (14)
\end{aligned}$$

The last expression takes the form of a boost along the x-axis with rapidity $\theta + \theta'$. Thus a composition of two boosts simply corresponds to adding the rapidities of the separate boosts. The velocity v'' is

$$v'' = c\tanh(\theta+\theta') = c\frac{\tanh\theta + \tanh\theta'}{1 + \tanh\theta\tanh\theta'} = \frac{v + v'}{1 + vv'/c^2}. \qquad (15)$$

As a check, note that this result reduces to the familiar Galilean velocity addition law $v'' = v + v'$ in the nonrelativistic limit $|v|, |v'| \ll c$.