Solutions to exercises for week 4

Exercise 1

(a) The equations of motion are given by the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} = 0 \qquad (i = 1, 2), \tag{1}$$

with $\mathcal{L} = \mathcal{L}(\phi_1, \partial_\mu \phi_1, \phi_2, \partial_\mu \phi_2)$. Thus

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \phi_i} &= \frac{\partial}{\partial \phi_i} \frac{1}{2} \sum_{j=1,2} \left[(\partial_\mu \phi_j)(\partial^\mu \phi_j) - m^2 \phi_j^2 \right] \\ &= \frac{1}{2} \sum_j (-m^2) \cdot 2\phi_j \delta_{ij} = -m^2 \phi_i. \end{aligned} \tag{2}$$

Furthermore,

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} &= \frac{1}{2} \sum_j \frac{\partial}{\partial(\partial_\mu \phi_i)} (\partial_\nu \phi_j)(\partial^\nu \phi_j) \\ &= \frac{1}{2} \frac{\partial}{\partial(\partial_\mu \phi_i)} (\partial_\nu \phi_i)(\partial^\nu \phi_i) = \frac{1}{2} g^{\nu\lambda} \frac{\partial}{\partial(\partial_\mu \phi_i)} (\partial_\nu \phi_i)(\partial_\lambda \phi_i) \\ &= \frac{1}{2} g^{\nu\lambda} [\delta^\mu_\nu \partial_\lambda \phi_i + \delta^\mu_\lambda \partial_\nu \phi_i] = g^{\nu\lambda} \delta^\mu_\nu \partial_\lambda \phi_i = \delta^\mu_\nu \partial^\nu \phi_i = \partial^\mu \phi_i. \end{aligned} \tag{3}$$

Here, to get the second expression on the third line we used the symmetry $g^{\nu\lambda} = g^{\lambda\nu}$ and a renaming of the dummy summation indices $\lambda \leftrightarrow \nu$. Inserting these results into the Euler-Lagrange equation gives

$$\left(\partial_\mu \partial^\mu + m^2 \right) \phi_i = 0 \qquad (i = 1, 2), \tag{4}$$

i.e. each of the two fields separately satisfies the Klein-Gordon equation.

(b) We have

$$\mathcal{L} \to \mathcal{L}' = \frac{1}{2} [(\partial_\mu \phi_1')(\partial^\mu \phi_1') + (\partial_\mu \phi_2')(\partial^\mu \phi_2')] - \frac{1}{2} m^2 [(\phi_1')^2 + (\phi_2')^2]. \tag{5}$$

The expressions inside the two square brackets both take the form

$$f_1' g_1' + f_2' g_2', \tag{6}$$

with $f_i = \partial_\mu \phi_i$ and $g_i = \partial^\mu \phi_i$ in the first bracket and $f_i = g_i = \phi_i$ in the second bracket (there's implicitly also a sum over μ in the first bracket, but we can consider each value of μ separately). Using

$$f_1' = f_1 \cos\alpha - f_2 \sin\alpha, \tag{7}$$

$$f_2' = f_1 \sin\alpha + f_2 \cos\alpha, \tag{8}$$

and similarly for the g functions, we get

$$\begin{aligned} f_1'g_1' + f_2'g_2' &= (f_1g_1 + f_2g_2)(\cos^2\alpha + \sin^2\alpha) + (f_1g_2 + f_2g_1 - f_1g_2 - f_2g_1)\cos\alpha\sin\alpha \\ &= f_1g_1 + f_2g_2, \end{aligned} \tag{9}$$

from which it follows that $\mathcal{L}' = \mathcal{L}$.

(c) The infinitesimal form is obtained by Taylor-expanding the general form of the transformation to first order in the parameter α. This gives

$$\phi_1 \rightarrow \phi_1' = \phi_1 - \alpha\phi_2, \tag{10}$$

$$\phi_2 \rightarrow \phi_2' = \alpha\phi_1 + \phi_2. \tag{11}$$

(d) We have

$$j^\mu = \sum_{i=1,2} \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\Delta\phi_i - \mathcal{J}^\mu. \tag{12}$$

Since $\mathcal{L}' = \mathcal{L}$, we can take $\mathcal{J}^\mu = 0$. The quantities $\Delta\phi_i$ can be read off from (10)-(11) as $\Delta\phi_1 = -\phi_2$ and $\Delta\phi_2 = \phi_1$. We found $\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}$ in (a). Putting it all together gives

$$j^\mu = \phi_1\partial^\mu\phi_2 - \phi_2\partial^\mu\phi_1. \tag{13}$$

Exercise 2

(a) Starting from

$$\mathcal{L} = (\partial_\mu\Phi^*)(\partial^\mu\Phi) - m^2\Phi^*\Phi, \tag{14}$$

we insert for Φ and Φ^* in terms of the real fields:

$$\Phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \tag{15}$$

$$\Phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \tag{16}$$

This gives

$$\begin{aligned} \mathcal{L} &= \left(\frac{1}{\sqrt{2}}\right)^2 [\partial_\mu(\phi_1 - i\phi_2)]\,[\partial^\mu(\phi_1 + i\phi_2)] - \left(\frac{1}{\sqrt{2}}\right)^2 m^2(\phi_1 - i\phi_2)(\phi_1 + i\phi_2) \\ &= \frac{1}{2}\Big[(\partial_\mu\phi_1)(\partial^\mu\phi_1) + (\partial_\mu\phi_2)(\partial^\mu\phi_2) + i(\partial_\mu\phi_1)(\partial^\mu\phi_2) - i(\partial_\mu\phi_2)(\partial^\mu\phi_1)\Big] \\ &- \frac{1}{2}m^2(\phi_1^2 + \phi_2^2 + i\phi_1\phi_2 - i\phi_1\phi_2). \end{aligned} \tag{17}$$

This is identical to the Lagrangian introduced in Exercise 1, because the imaginary terms in the mass part ($\propto m^2$) obviously cancel, and so do the imaginary terms in the kinetic part. This latter fact can e.g. be seen as follows:

$$\begin{aligned} (\partial_\mu\phi_1)(\partial^\mu\phi_2) &= (g_{\mu\nu}\partial^\nu\phi_1)(g^{\mu\lambda}\partial_\lambda\phi_2) = g_{\mu\nu}g^{\mu\lambda}(\partial^\nu\phi_1)(\partial_\lambda\phi_2) = \delta_\nu^\lambda(\partial^\nu\phi_1)(\partial_\lambda\phi_2) \\ &= (\partial^\nu\phi_1)(\partial_\nu\phi_2). \end{aligned} \tag{18}$$

Here we used that the matrix "g with upper indices" is by definition the inverse of the matrix "g with lower indices" (the metric tensor) and therefore $g_{\nu\mu}g^{\mu\lambda} = \delta_\nu^\lambda$.

(b) By using the expression (14) for $\mathcal{L}$, and treating Φ and Φ^* as independent fields, one finds easily

$$\frac{\partial \mathcal{L}}{\partial \Phi^*} = -m^2\Phi, \tag{19}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^*)} = \partial^\mu \Phi. \tag{20}$$

Inserting these results into the Euler-Lagrange equation for Φ^* gives

$$(\partial_\mu \partial^\mu + m^2)\Phi = 0. \tag{21}$$

As in Exercise 1, we arrive at the Klein-Gordon equation, but now it is a complex equation since Φ is complex. By considering the real and imaginary parts of this single complex equation, one can reexpress it as two real equations, which are the Klein-Gordon equations for the real fields ϕ_1 and ϕ_2 found in 1(a).

(c) This follows easily because, in each term in $\mathcal{L}$, the number of factors involving Φ equals the number of factors involving Φ^*, and therefore the α dependence in $\mathcal{L}'$ cancels since $e^{-i\alpha}e^{i\alpha} = 1$.

(d) Expanding to first order in α gives

$$\Phi \rightarrow \Phi' = \Phi(1 + i\alpha) = \Phi + i\alpha\Phi, \tag{22}$$

$$\Phi^* \rightarrow \Phi'^* = \Phi^*(1 - i\alpha) = \Phi^* - i\alpha\Phi^*. \tag{23}$$

(e) We have

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\Delta\Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^*)}\Delta\Phi^* - \mathcal{J}^\mu. \tag{24}$$

Since $\mathcal{L}$ is invariant we can take $\mathcal{J}^\mu = 0$. From the infinitesimal form in 2(d) we find

$$\Delta\Phi = i\Phi, \tag{25}$$

$$\Delta\Phi^* = -i\Phi^*. \tag{26}$$

Using also (20) and its c.c. (note that $\mathcal{L}$ is real) we find

$$j^\mu = i\left[\Phi\partial^\mu\Phi^* - \Phi^*\partial^\mu\Phi\right]. \tag{27}$$

By inserting (15)-(16) one can easily verify that all terms involving only one of the real fields cancel, leaving the expression (13).