Solution to week 5 exercises

Problem 3.6.1

1) We are given the nonrelativistic Lagrangian density

$$\mathcal{L} = \frac{1}{2} i\hbar(\psi^* \dot{\psi} - \psi \dot{\psi}^*) - \frac{\hbar^2}{2m}(\nabla\psi^*) \cdot (\nabla\psi) \tag{1}$$

where $\dot{\psi} \equiv \partial\psi/\partial t$. We can either write $\psi = \psi_1 + i\psi_2$ and treat ψ_1 and ψ_2 as independent fields, or we can treat ψ and ψ^* as independent fields. Choosing the latter, the Euler-Lagrange equations can be written[1]

$$\frac{\partial\mathcal{L}}{\partial\psi} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{\psi}} - \nabla \cdot \frac{\partial\mathcal{L}}{\partial(\nabla\psi)} = 0 \tag{2}$$

and

$$\frac{\partial\mathcal{L}}{\partial\psi^*} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{\psi}^*} - \nabla \cdot \frac{\partial\mathcal{L}}{\partial(\nabla\psi^*)} = 0, \tag{3}$$

which are each other's c.c. Here we have used the notation $\frac{\partial\mathcal{L}}{\partial(\nabla\psi)}$ to denote the 3-vector whose (say) x-component is $\frac{\partial\mathcal{L}}{\partial(\frac{\partial\psi}{\partial x})}$. We have

$$\frac{\partial\mathcal{L}}{\partial(\nabla\psi^*)} = -\frac{\hbar^2}{2m}\nabla\psi \ , \qquad \frac{\partial\mathcal{L}}{\partial(\dot{\psi}^*)} = -\frac{1}{2}i\hbar\psi \ , \qquad \frac{\partial\mathcal{L}}{\partial\psi^*} = \frac{1}{2}i\hbar\dot{\psi} \ . \tag{4}$$

Putting this into (3), we obtain

$$i\hbar\dot{\psi} + \frac{\hbar^2}{2m}\nabla^2\psi \quad = \quad 0 \ , \tag{5}$$

which we recognize as the Schrödinger equation for a free particle of mass m.

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[1] These follow easily from the form of the E-L equations we have used earlier, i.e.

$$\frac{\partial\mathcal{L}}{\partial\psi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = 0$$

and its complex conjugate, as the factors of c in the two ∂_0's cancel in the $\mu = 0$ term in the sum.

2) The Hamiltonian density is given by

$$\mathcal{H} = \pi_\psi \dot{\psi} + \pi_{\psi^*} \dot{\psi}^* - \mathcal{L} \tag{6}$$

where

$$\pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \frac{1}{2} i\hbar \psi^*, \quad \pi_{\psi^*} = \frac{\partial \mathcal{L}}{\partial \dot{\psi}^*} = -\frac{1}{2} i\hbar \psi. \tag{7}$$

Thus

$$\begin{aligned} \mathcal{H} &= \frac{1}{2} i\hbar \psi^* \dot{\psi} - \frac{1}{2} i\hbar \psi \dot{\psi}^* - \left[\frac{1}{2} i\hbar (\psi^* \dot{\psi} - \psi \dot{\psi}^*) - \frac{\hbar^2}{2m} (\nabla \psi^*) \cdot (\nabla \psi) \right] \\ &= \frac{\hbar^2}{2m} (\nabla \psi^*) \cdot (\nabla \psi) \ . \end{aligned} \tag{8}$$

[The Hamiltonian H is therefore

$$H = \int d^3x \ \mathcal{H} = \int d^3x \ \frac{\hbar^2}{2m} (\nabla \psi^*) \cdot (\nabla \psi). \tag{9}$$

Integrating by parts and assuming that the Hamiltonian density vanishes at infinity, we obtain the familiar form

$$H = \int d^3x \ \psi^* \left(-\frac{\hbar^2}{2m} \right) \nabla^2 \psi \tag{10}$$

for the total energy.]

3) The global phase transformation reads

$$\psi \rightarrow e^{i\alpha} \psi \ , \tag{11}$$
$$\psi^* \rightarrow e^{-i\alpha} \psi^* \ . \tag{12}$$

Since α is a constant and every term in $\mathcal{L}$ has one factor of ψ and one factor of ψ^*, it is clear that $\mathcal{L}$ is invariant.

4) Since $\mathcal{L}$ is invariant, we have the continuity equation

$$\partial_\mu j^\mu = 0 \ , \tag{13}$$

where

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \Delta \psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)} \Delta \psi^*. \tag{14}$$

From Eqs. (11) and (12), we obtain

$$\Delta \psi = i\psi \ , \tag{15}$$
$$\Delta \psi^* = -i\psi^* \ . \tag{16}$$

The terms multiplying $\Delta\psi$ and $\Delta\psi^*$ in (14) can be found from the first two terms in (4) and their c.c.'s. This gives

$$\begin{aligned} j^0 &= c(\psi^*\psi) &(17)\\ \vec{j} &= \frac{\hbar}{2mi}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)\ . &(18)\end{aligned}$$

Actually, here we have divided both j^0 and $\vec{j}$ by $\hbar$ as that gives a more common normalization. The continuity equation can be written

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{j} = 0, \tag{19}$$

where $\rho = j^0/c = \psi^*\psi$ can be interpreted as a probability density, while $\vec{j}$ is the probability current density. The continuity equation is a local conservation law for probability, valid at each point in space: a change in probability inside an infinitesimal volume surrounding the point can only occur through net transport of probability into or out of the volume through its boundaries. The continuity equation can also be obtained by multiplying the Schrödinger equation (5) with ψ^* and multiplying its complex conjugate by ψ, and subtracting the two.

5) This is done by letting

$$\partial_\mu \to \partial_\mu + iqA_\mu/\hbar \quad (\equiv D_\mu) \tag{20}$$

in the Lagrangian density (1). Using $A^\mu = (A^0, \vec{A})$ with $A^0 = \varphi/c$, we get $A_\mu = (\varphi/c, -\vec{A})$. Thus

$$\begin{aligned} \partial_t &\to \partial_t + iq\varphi/\hbar, &(21)\\ \nabla &\to \nabla - iq\vec{A}/\hbar. &(22)\end{aligned}$$

The factors of i here imply that we should take care to do this transformation on the following rewritten form of (1):

$$\mathcal{L} = \frac{1}{2}i\hbar[\psi^*(\partial_t\psi) - \psi(\partial_t\psi)^*] - \frac{\hbar^2}{2m}(\nabla\psi)^*\cdot(\nabla\psi), \tag{23}$$

where the last two complex conjugations include also the differential operators acting on the function being complex conjugated; this ensures that also the new Lagrangian density resulting from the transformation is real-valued. It reads

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}i\hbar[\psi^*((\partial_t + iq\varphi/\hbar)\psi) - \psi((\partial_t + iq\varphi/\hbar)\psi)^*] - \frac{\hbar^2}{2m}((\nabla - iq\vec{A}/\hbar)\psi)^*\cdot((\nabla - iq\vec{A}/\hbar)\psi)\\ &= \frac{1}{2}i\hbar\left[\psi^*(\partial_t + iq\varphi/\hbar)\psi - \psi(\partial_t - iq\varphi/\hbar)\psi^*\right] - \frac{\hbar^2}{2m}\left[(\nabla + iq\vec{A}/\hbar)\psi^*\right]\cdot\left[(\nabla - iq\vec{A}/\hbar)\psi\right]. \quad (24)\end{aligned}$$

[It is an instructive exercise to construct the Hamiltonian H following the same steps as in 2) and show that one gets the expected result

$$H = \int d^3x\, \psi^*\left[\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - q\vec{A}\right)^2 + q\varphi\right]\psi. \tag{25}$$

]

Problem 3.6.3

[Note: In this problem the constants c, μ_0, and ε_0 that appear in electromagnetic theory are set equal to 1 (this simplification is consistent with the relation $\varepsilon_0\mu_0 = 1/c^2$).]

The Lagrangian for the electromagnetic field with sources is given by

$$\mathcal{L} \;=\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu \tag{26}$$

where $j^\mu = (\rho, \mathbf{j})$. $F^{\mu\nu}$ is gauge invariant (we showed this in the lectures) and therefore $F_{\mu\nu}$ is too. Hence the first term in $\mathcal{L}$ is gauge invariant. In the second term, j^μ is gauge invariant because its components are physically observable (they appear in the equations of motion, i.e. the Maxwell equations), while A_μ transforms as $A_\mu \to A_\mu - \partial_\mu\alpha$. This gives

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} + j^\mu\partial_\mu\alpha. \tag{27}$$

Thus $\mathcal{L}' \neq \mathcal{L}$. But the difference $\mathcal{L}' - \mathcal{L}$ is a total divergence $\partial_\mu(j^\mu\alpha)$, since

$$\partial_\mu(j^\mu\alpha) = \alpha\partial_\mu j^\mu + j^\mu\partial_\mu\alpha = j^\mu\partial_\mu\alpha = \mathcal{L}' - \mathcal{L} \tag{28}$$

where we used that j^μ satisfies $\partial_\mu j^\mu = 0$, the continuity equation for electric charge. Therefore $\mathcal{L}'$ and $\mathcal{L}$ are equivalent Lagrangian densities: they give the same equations of motion. We can summarize the situation by saying that $\mathcal{L}$ is gauge invariant up to a total divergence. (In contrast, we will reserve the phrase "not gauge invariant" for terms that under a gauge transformation give rise to a change in $\mathcal{L}$ that is not a total divergence, so that the equations of motion are affected. This would be the case if we e.g. tried to include a term $\propto A_\mu A^\mu$ in the Lagrangian density. Such a term is therefore not allowed in the Lagrangian density for the electromagnetic field.)

The Euler-Lagrange equations are

$$\frac{\partial\mathcal{L}}{\partial A_\sigma} - \partial_\gamma\frac{\partial\mathcal{L}}{\partial(\partial_\gamma A_\sigma)} = 0. \tag{29}$$

Let us first rewrite $\mathcal{L}$ so that derivatives of the gauge field have all indices downstairs:

$$\mathcal{L} \;=\; -\frac{1}{4}F_{\mu\nu}g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta} - j^\mu A_\mu, \tag{30}$$

This gives

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial(\partial_\gamma A_\sigma)} &= -\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}\frac{\partial}{\partial(\partial_\gamma A_\sigma)}F_{\mu\nu}F_{\alpha\beta}\\ &= -\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}\left[F_{\mu\nu}\frac{\partial}{\partial(\partial_\gamma A_\sigma)}F_{\alpha\beta} + F_{\alpha\beta}\frac{\partial}{\partial(\partial_\gamma A_\sigma)}F_{\mu\nu}\right]\end{aligned} \tag{31}$$

In the first term, rename the summation indices as $\mu \leftrightarrow \alpha$, $\nu \leftrightarrow \beta$ and use the symmetry $g^{\lambda\kappa} = g^{\kappa\lambda}$. This shows that the first term is identical to the second. Thus

$$\frac{\partial \mathcal{L}}{\partial(\partial_\gamma A_\sigma)} = -\frac{1}{2} g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} \frac{\partial}{\partial(\partial_\gamma A_\sigma)} F_{\mu\nu}. \tag{32}$$

Now, since

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{33}$$

we get

$$\frac{\partial}{\partial(\partial_\gamma A_\sigma)} F_{\mu\nu} = \delta^\gamma_\mu \delta^\sigma_\nu - \delta^\sigma_\mu \delta^\gamma_\nu. \tag{34}$$

Thus

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial(\partial_\gamma A_\sigma)} &= -\frac{1}{2} g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} (\delta^\gamma_\mu \delta^\sigma_\nu - \delta^\sigma_\mu \delta^\gamma_\nu) \\ &= -\frac{1}{2} F_{\alpha\beta} (g^{\gamma\alpha} g^{\sigma\beta} - g^{\sigma\alpha} g^{\gamma\beta}) = -\frac{1}{2} (F^{\gamma\sigma} - F^{\sigma\gamma}) \\ &= -F^{\gamma\sigma}. \end{aligned} \tag{35}$$

Furthermore,

$$\frac{\partial \mathcal{L}}{\partial A_\sigma} = -j^\sigma. \tag{36}$$

The Euler-Lagrange equations then become

$$\partial_\gamma F^{\gamma\sigma} = j^\sigma. \tag{37}$$

Now let us use

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{38}$$

to rewrite (37) in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$. For $\sigma = 0$ we obtain

$$\partial_\gamma F^{\gamma 0} = \rho\ . \tag{39}$$

Since F^{00} vanishes this gives

$$\partial_i F^{i0} = \rho\ , \tag{40}$$

or

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \rho, \tag{41}$$

i.e.

$$\underline{\underline{\nabla \cdot \mathbf{E} = \rho}}\ . \tag{42}$$

Similarly, for $\sigma = i$ we obtain

$$\partial_0 F^{0i} + \partial_j F^{ji} \quad = \quad j^i \,. \tag{43}$$

Using the different components of F^{ij}, this equation can be seen to be the i'th component of the vector equation

$$\underline{\underline{-\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} = \mathbf{j}}} \,. \tag{44}$$

Eqs. (42) and (44) are the inhomogeneous Maxwell equations. The homogeneous equations simply guarantee that we can express $\mathbf{E}$ and $\mathbf{B}$ in terms of a scalar potential φ and a vector potential $\mathbf{A}$.