Solution for week 6 exercises

In these exercises we have set $\hbar = c = 1$.

Exercise 1

(a) The Lagrangian density is

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi. \tag{1}$$

The Euler-Lagrange equation for $\bar{\psi}$ is

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = 0. \tag{2}$$

We have

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \bar{\psi}} &= (i\gamma^\mu \partial_\mu - m)\psi, & (3) \\ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} &= 0. & (4) \end{aligned}$$

Therefore (2) becomes

$$(i\gamma^\mu \partial_\mu - m)\psi = 0, \tag{5}$$

which is the Dirac equation.

(b) We have

$$\begin{aligned} \pi_\psi &= \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = \bar{\psi} i \gamma^0, & (6) \\ \pi_{\bar{\psi}} &= \frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\psi})} = 0. & (7) \end{aligned}$$

Because of the latter result, there is no term involving $\partial_0 \bar{\psi}$ in $\mathcal{H}$. Thus

$$\begin{aligned} \mathcal{H} &= \pi_\psi \partial_0 \psi - \mathcal{L} = \bar{\psi} i \gamma^0 - \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \\ &= -\bar{\psi}(i\gamma^j \partial_j - m)\psi = -\psi^\dagger \gamma^0 (i\gamma^j \partial_j - m)\psi \\ &= \psi^\dagger(-i\gamma^0 \gamma^j \partial_j + \gamma^0 m)\psi \\ &= \psi^\dagger(-i\vec{\alpha} \cdot \nabla + \beta m)\psi. & (8) \end{aligned}$$

To get the last line we first used $\gamma^0 = \beta$ and $\vec{\gamma} = \beta\vec{\alpha}$ as defined in the lectures. Then multiplying the latter equation by β from the left and using $\beta^2 = I$ gives $\alpha = \beta\vec{\gamma}$.

(c) The Lagrangian density transforms as

$$\mathcal{L} \to \bar{\psi}e^{i\lambda}(i\gamma^\mu\partial_\mu - m)e^{-i\lambda}\psi. \tag{9}$$

Both the mass and kinetic terms are invariant due to $e^{i\lambda}e^{-i\lambda} = 1$. In the mass term this result can be found immediately. In the kinetic terms one first has to move $e^{-i\lambda}$ to the left so it stands next to $e^{-i\lambda}$. This can be done without further ado because $e^{-i\lambda}$ commutes with both ∂_μ and γ^μ (because $e^{-i\lambda}$ is respectively independent of space-time coordinates and just a number).

From the infinitesimal form of the transformation we identify

$$\begin{aligned} \Delta\psi &= -i\psi, && (10) \\ \Delta\bar{\psi} &= i\bar{\psi}. && (11) \end{aligned}$$

However, because of (4) there is no contribution to j^μ from $\Delta\bar{\psi}$. Thus

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\Delta\psi = \bar{\psi}i\gamma^\mu\cdot(-i\psi) = \bar{\psi}\gamma^\mu\psi. \tag{12}$$

This gives

$$\begin{aligned} j^0 &= \bar{\psi}\gamma^0\psi = \psi^\dagger(\gamma^0)^2\psi = \psi^\dagger\psi, && (13) \\ \vec{j} &= \bar{\psi}\vec{\gamma}\psi = \psi^\dagger\gamma^0\vec{\gamma}\psi = \psi^\dagger\vec{\alpha}\psi. && (14) \end{aligned}$$

We found the same results for j^0 ($= \rho$) and $\vec{j}$ in the lectures using a different method.

Excercise 2

(a) We have

$$\begin{aligned} (\gamma^0)^\dagger &= \gamma^0, && (15) \\ (\gamma^i)^\dagger &= -\gamma^i. && (16) \end{aligned}$$

The first result follows because β is hermitian. The second result follows because also the α-matrices are hermitian, and they anticommute with β. These results can be written in a unified way as

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0. \tag{17}$$

Now consider

$$(\gamma^5)^\dagger = -i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger\ . \tag{18}$$

Inserting (17), we find

$$\begin{aligned}
(\gamma^5)^\dagger &= -i(\gamma^0\gamma^3\gamma^0)(\gamma^0\gamma^2\gamma^0)(\gamma^0\gamma^1\gamma^0)(\gamma^0\gamma^0\gamma^0) \\
&= -i\gamma^0\gamma^3\gamma^2\gamma^1 \\
&= -i\gamma^0\gamma^1\gamma^3\gamma^2 \\
&= i\gamma^0\gamma^1\gamma^2\gamma^3 \\
&= \gamma^5 \, . \qquad (19)
\end{aligned}$$

Thus γ^5 is hermitean. Furthermore,

$$\begin{aligned}
(\gamma^5)^2 &= (i\gamma^0\gamma^1\gamma^2\gamma^3)^2 \\
&= -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 \\
&= +\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3\gamma^0\gamma^1\gamma^2 \\
&= -\gamma^0\gamma^1\gamma^2\gamma^0\gamma^1\gamma^2 \\
&= -\gamma^0\gamma^1\gamma^2\gamma^2\gamma^0\gamma^1 \\
&= +\gamma^0\gamma^1\gamma^0\gamma^1 \\
&= -\gamma^0\gamma^1\gamma^1\gamma^0 \\
&= +\gamma^0\gamma^0 \\
&= 1 \, . \qquad (20)
\end{aligned}$$

Next consider $\gamma^5\gamma^\mu$. Since $\mu = 0, 1, 2$ or 3, γ^μ commutes with one of the matrices in γ^5 and anticommutes with the remaining three. We therefore generate a total sign $(-1)^3 = -1$ as we move γ^μ to the left :

$$\begin{aligned}
\gamma^5\gamma^\mu &= (i\gamma^0\gamma^1\gamma^2\gamma^3)\gamma^\mu \\
&= -\gamma^\mu(i\gamma^0\gamma^1\gamma^2\gamma^3) \\
&= -\gamma^\mu\gamma^5 \, . \qquad (21)
\end{aligned}$$

In other words, γ^5 anticommutes with γ^μ:

$$\{\gamma^5, \gamma^\mu\} = 0 \, . \qquad (22)$$

(b) We have

$$\begin{aligned}
\gamma^\mu e^{i\gamma^5\lambda} &= \gamma^\mu \sum_{n=0}^{\infty} \frac{(i\gamma^5\lambda)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(i\lambda)^n}{n!}\gamma^\mu(\gamma^5)^n \\
&= \sum_{n=0}^{\infty} \frac{(i\lambda)^n}{n!}(-1)^n(\gamma^5)^n\gamma^\mu \\
&= \sum_{n=0}^{\infty} \frac{(-i\gamma^5\lambda)^n}{n!}\gamma^\mu \\
&= e^{-i\gamma^5\lambda}\gamma^\mu. \qquad (23)
\end{aligned}$$

The key step of this proof is the transition between the second and third line, where we move γ^μ all the way to the right, past n factors of γ^5. Thus n anticommutations are done, giving a total sign $(-1)^n$. (Note that we didn't need to make use of $(\gamma^5)^2 = 1$ in the proof.)

(c) To answer this question we investigate how $\mathcal{L}$ transforms under the chiral transformation. First consider the kinetic terms, which transform as

$$\begin{aligned} \bar{\psi}\gamma^\mu\partial_\mu\psi &\rightarrow \bar{\psi}e^{i(\gamma^5)^\dagger\lambda}\gamma^\mu\partial_\mu e^{i\gamma^5\lambda}\psi \\ &= \bar{\psi}e^{i\gamma^5\lambda}\gamma^\mu e^{i\gamma^5\lambda}\partial_\mu\psi \\ &= \bar{\psi}e^{i\gamma^5\lambda}e^{-i\gamma^5\lambda}\gamma^\mu\partial_\mu\psi \\ &= \bar{\psi}\gamma^\mu\partial_\mu\psi. \end{aligned} \tag{24}$$

Thus the kinetic part of $\mathcal{L}$ is invariant. The mass term transforms as

$$\begin{aligned} \bar{\psi}\psi &\rightarrow \bar{\psi}e^{i(\gamma^5)^\dagger\lambda}e^{i\gamma^5\lambda}\psi \\ &= \bar{\psi}e^{2i\gamma^5\lambda}\psi\ . \end{aligned} \tag{25}$$

Thus the mass term is not invariant. Furthermore, considering an infinitesimal transformation, the change in the mass term can not be written as a total divergence either. It follows that the chiral transformation is a symmetry of $\mathcal{L}$ only if the mass term is absent, i.e. if $m = 0$. To find the conserved current j^μ in this massless case, we use the infinitesimal transformation to identify

$$\Delta\psi = i\gamma^5\psi\ , \tag{26}$$

$$\Delta\bar{\psi} = i\bar{\psi}\gamma^5\ . \tag{27}$$

However, as in 1(c), because of (4) there is no contribution to j^μ from the $\Delta\bar{\psi}$ term. Thus

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\Delta\psi = \bar{\psi}i\gamma^\mu \cdot i\gamma^5\psi = -\bar{\psi}\gamma^\mu\gamma^5\psi. \tag{28}$$

A final remark: One thing that might at first sight seem a bit odd about the chiral transformation is that both exponents have the same sign. Let us see how this comes about. The chiral transformation of ψ is defined as

$$\psi \rightarrow e^{i\gamma^5\lambda}\ \psi. \tag{29}$$

Taking the adjoint of each side of this transformation, it follows that $\psi^\dagger$ should transform as

$$\psi^\dagger \rightarrow \psi^\dagger e^{-i(\gamma^5)^\dagger\lambda}. \tag{30}$$

Now multiply both sides of this transformation by γ^0 from the right to get the transformation of $\bar{\psi}$:

$$\bar{\psi} \rightarrow \psi^\dagger e^{-i(\gamma^5)^\dagger\lambda}\gamma^0. \tag{31}$$

To make $\bar{\psi}$ appear also on the right side of this transformation, we can move γ^0 past the exponential, which changes the sign in the exponent, thus giving

$$\bar{\psi} \rightarrow \bar{\psi}\ e^{i(\gamma^5)^\dagger\lambda}. \tag{32}$$