# Solution to week 7 exercise

(a) We use the Pauli matrix identity

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k \tag{1}$$

where $\epsilon_{ijk}$ is the totally antisymmetric Levi-Civita tensor with $\epsilon_{123} = +1$ and there is a sum over $k = 1, 2, 3$ in the last term. This gives

$$\{\sigma_i, \sigma_j\} \;=\; 2\delta_{ij} + i\underbrace{(\epsilon_{ijk} + \epsilon_{jik})}_{=0}\sigma_k = 2\delta_{ij}. \tag{2}$$

From this one easily sees that

$$\{\gamma^\mu, \gamma^\nu\} = \left\{ \begin{array}{rl} 2 & \text{if } \mu = \nu = 0 \\ -2 & \text{if } \mu = \nu = 1, 2 \\ 0 & \text{if } \mu \neq \nu \end{array} \right\} = 2g^{\mu\nu}. \tag{3}$$

(b) The electric field vanishes for all three configurations since $A^0 = 0$ and $\frac{\partial \vec{A}}{\partial t} = 0$. Moreover,

$$\vec{B} \;=\; \nabla \times \vec{A}\,.$$

In the first case, we obtain

$$\vec{B} \;=\; \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & Bx & 0 \end{vmatrix} = B\vec{k}\,.$$

The other gauge fields yield the same constant magnetic field pointing along the $z$-axis.

(c) With the given gauge field the Dirac equation becomes

$$(i\gamma^0\partial_t + i\gamma^1\partial_x + i\gamma^2\partial_y - q\gamma^2 A_2 - m)\psi = 0. \tag{4}$$

Using $A_2 = -A^2 = -Bx$ and the expressions for the $\gamma$ matrices we get

$$(i\sigma_3\partial_t - \sigma_2\partial_x + \sigma_1\partial_y - iqBx\sigma_1 - m)\psi = 0. \tag{5}$$

Inserting the Pauli matrices, this becomes

$$\left[\begin{pmatrix} i\partial_t & 0 \\ 0 & -i\partial_t \end{pmatrix} + \begin{pmatrix} 0 & i\partial_x \\ -i\partial_x & 0 \end{pmatrix} + \begin{pmatrix} 0 & \partial_y \\ \partial_y & 0 \end{pmatrix} + \begin{pmatrix} 0 & -iqBx \\ -iqBx & 0 \end{pmatrix} + \begin{pmatrix} -m & 0 \\ 0 & -m \end{pmatrix}\right]\psi$$
$$= \begin{pmatrix} i\partial_t - m & i\partial_x + \partial_y - iqBx \\ -i\partial_x + \partial_y - iqBx & -i\partial_t - m \end{pmatrix}\psi = 0. \tag{6}$$

(d) Since $\partial_y$ appears in this matrix but $y$ does not, the $y$-dependence of the eigenstates will have a plane-wave form, i.e. $\propto e^{ip_y y}$.

(e) Inserting the expression for $\psi$ into the Dirac equation and carrying out the differentiations, we obtain

$$\begin{pmatrix} E - m & -\xi_+ \\ \xi_- & -E - m \end{pmatrix}\begin{pmatrix} f(x) \\ g(x) \end{pmatrix} = 0\ . \tag{7}$$

(f) The set of equations can be written as

$$(E - m)f(x) - \xi_+ g(x) = 0\ , \tag{8}$$
$$\xi_- f(x) - (E + m)g(x) = 0\ . \tag{9}$$

The second equation gives

$$g(x) = \frac{1}{E + m}\xi_- f(x). \tag{10}$$

Inserting this into the first equation, we obtain

$$(E^2 - m^2)f(x) = \xi_+\xi_- f(x)\ . \tag{11}$$

(g) Multiplying out, simplifying, rearranging, and dividing by $2m$ for convenience, we find

$$\frac{1}{2m}\left[-\partial_x^2 + (qB)^2\left(x - \frac{p_y}{qB}\right)^2\right]f(x) = \frac{E^2 - m^2 + qB}{2m}f(x)\ . \tag{12}$$

This takes the form of the eigenvalue equation $H\psi_n = \Omega_n\psi_n$ for a harmonic oscillator centered at $x = p_y/qB$. We identify $m\omega^2/2 = (qB)^2/2m$ (so $\omega = qB/m$), $\Omega_n = (E^2 - m^2 + qB)/2m$, and $\psi_n(x - p_y/qB) = f(x)$. From $\Omega_n = \omega(1/2 + n)$ we obtain

$$E^2 = m^2 + 2qBn \quad (n = 0, 1, 2, \ldots). \tag{13}$$

(h) The function $g(x)$ is found using Eq. (10).