Solutions for week 8 exercises

Exercise 4.3.1

We can write

$$\begin{aligned}(\boldsymbol{\sigma}\cdot\boldsymbol{A})(\boldsymbol{\sigma}\cdot\boldsymbol{B}) &= A_i\sigma_i B_j\sigma_j \\ &= A_i B_j\sigma_i\sigma_j\ ,\end{aligned}\tag{1}$$

where we are summing over i and j. Using

$$\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k\ ,\tag{2}$$

where ϵ_{ijk} is the totally antisymmetric Levi-Civita tensor with $\epsilon_{123}=+1$, we can write

$$\begin{aligned}(\boldsymbol{\sigma}\cdot\boldsymbol{A})(\boldsymbol{\sigma}\cdot\boldsymbol{B}) &= A_iB_j(\delta_{ij}+i\epsilon_{ijk}\sigma_k) \\ &= A_iB_i + iA_iB_j\epsilon_{ijk}\sigma_k \\ &= A_iB_i + i\epsilon_{kij}\sigma_kA_iB_j\ ,\end{aligned}\tag{3}$$

where in the last line we used $\epsilon_{ijk} = -\epsilon_{kji} = -(-\epsilon_{kij}) = \epsilon_{kij}$. Using

$$(\mathbf{A}\times\mathbf{B})_k = \epsilon_{kij}A_iB_j\ ,\tag{4}$$

we obtain

$$\begin{aligned}(\boldsymbol{\sigma}\cdot\boldsymbol{A})(\boldsymbol{\sigma}\cdot\boldsymbol{B}) &= A_iB_i + i\sigma_k(\mathbf{A}\times\mathbf{B})_k \\ &= \underline{\underline{(\mathbf{A}\cdot\mathbf{B}) + i\boldsymbol{\sigma}\cdot(\mathbf{A}\times\mathbf{B})}}\ .\end{aligned}\tag{5}$$

Exercise 5.6.1

1) Using $\hat{x} = \frac{l}{\sqrt{2}}(\hat{a}+\hat{a})$, we get

$$\begin{aligned}\hat{H}_1 &= \frac{\lambda}{24}\hat{x}^4 \\ &= \underline{\underline{\frac{\lambda l^4}{96}(\hat{a}+\hat{a}^\dagger)^4}}\ .\end{aligned}\tag{6}$$

Expanding out $(\hat{a}+\hat{a}^\dagger)^4$, there are 6 terms with two creation operators and two annihilation operators. To write a term in normal-ordered form we need to manipulate it (using $\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1$) so that all creation operators stand to the left of all annihilation operators. For example,

$$\begin{aligned}
\hat{a}\hat{a}\hat{a}^\dagger\hat{a}^\dagger &= a(1+\hat{a}^\dagger\hat{a})\hat{a}^\dagger \\
&= \hat{a}\hat{a}^\dagger + \hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger \\
&= (1+\hat{a}^\dagger\hat{a}) + (1+\hat{a}^\dagger\hat{a})(1+\hat{a}^\dagger\hat{a}) \\
&= 2 + 3\hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} \\
&= 2 + 3\hat{a}^\dagger\hat{a} + \hat{a}^\dagger(1+\hat{a}^\dagger\hat{a})\hat{a} \\
&= \underline{\underline{2 + 4\hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}}}\ . \qquad (7)
\end{aligned}$$

Calculating the other terms in the same manner, one finds

$$\begin{aligned}
\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger &= \underline{\underline{1 + 3\hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}}}\ , \qquad (8) \\
\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} &= \underline{\underline{\hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}}}\ , \qquad (9) \\
\hat{a}^\dagger\hat{a}\hat{a}\hat{a}^\dagger &= \underline{\underline{2\hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}}}\ , \qquad (10) \\
\hat{a}\hat{a}^\dagger\hat{a}^\dagger\hat{a} &= \underline{\underline{2\hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}}}\ , \qquad (11) \\
\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} &= \underline{\underline{\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}}}\ . \qquad (12)
\end{aligned}$$

2) The first-order shift in the ground state energy is given by

$$E_0^{(1)} = \langle 0|\hat{H}_1|0\rangle = \frac{\lambda l^4}{96}\langle 0|(\hat{a}+\hat{a}^\dagger)^4|0\rangle. \qquad (13)$$

A necessary (but not sufficient) criterion for a term (call it A) in $(\hat{a}+\hat{a}^\dagger)^4$ to have a nonzero vacuum expectation value $\langle 0|A|0\rangle$ is that A must contain an equal number of creation and annihilation operators (hence 2 of each kind, since the total number of operators in each term is 4). If this is not the case, the state $A|0\rangle$ cannot contain a component with 0 particles and hence it cannot have a nonzero overlap with $|0\rangle$ (since two states with different numbers of particles have zero overlap). Thus only the 6 terms identified in 1) need to be considered. Since we have written them on normal-ordered form, the only contribution to the vacuum expectation value can come from constants (as the vacuum expectation value of a normal-ordered product of operators is 0 because of $\hat{a}|0\rangle = 0$ and its adjoint $\langle 0|\hat{a}^\dagger = 0$). Only Eqs. (7) and (8) contain constants. This yields

$$E_0^{(1)} = \underline{\underline{3\frac{\lambda l^4}{96}}}\ .$$

A final remark: If the only goal here had been to calculate vacuum expectation values (which it wasn't; another goal was to give you practice in manipulating creation and annihilation operators) it wouldn't have been necessary to normal-order all 6 terms in 1). For example, the left-hand-sides of Eqs. (9)-(12) can immediately be seen to have vanishing vacuum expectation values, because their rightmost operator is an $\hat{a}$, or their leftmost operator is an $\hat{a}^\dagger$, or both. Also, for the same reason the manipulation on the line immediately before Eq. (7) wouldn't have been necessary.